

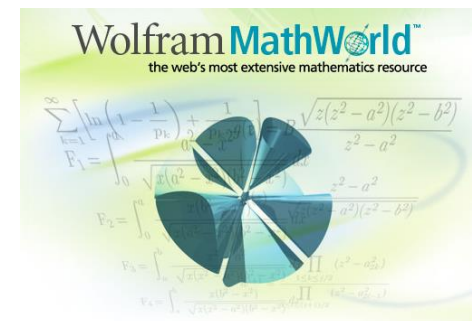
Summary

- ▶ Definition: Graph
- ▶ Related Definitions
- ▶ Applications

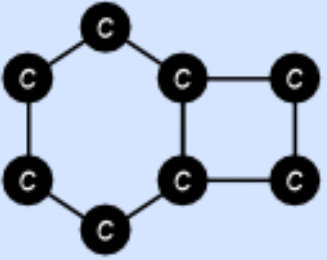
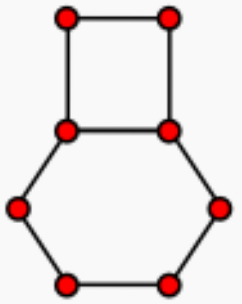
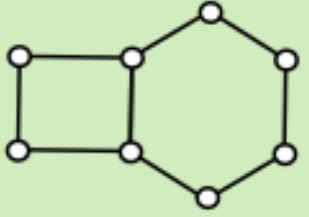

Definition: **Graph**

- ▶ A **graph** is a collection of **points** and **lines** connecting some (possibly empty) subset of them.
- ▶ The points of a graph are most commonly known as **graph vertices**, but may also be called “nodes” or simply “points.”
- ▶ The lines connecting the vertices of a graph are most commonly known as **graph edges**, but may also be called “arcs” or “lines.”

<http://mathworld.wolfram.com/>



What's in a name?

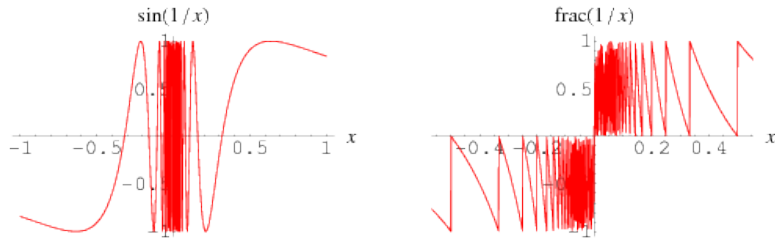
CHEMISTRY	SOCIAL NETWORKS	BIOLOGY	MATH
			<p>THEY LOOK THE SAME TO ME.</p> <p>LET'S CALL IT A GRAPH.</p> 
<p>BENZOCYCLOBUTADIENE</p> <p>C CARBON ATOMS</p> <p>— σ-ELECTRON BONDS</p>	<p>spikedmath.com © 2011</p> <p>● INDIVIDUALS</p> <p>— FRIENDSHIPS</p>	<p>PPI (SUB)NETWORK OF A SIMPLE ORGANISM</p> <p>○ PROTEINS</p> <p>— INTERACTIONS</p>	

"MATHEMATICS IS THE ART OF GIVING THE SAME NAME TO DIFFERENT THINGS."
JULES HENRI POINCARÉ (1854-1912)

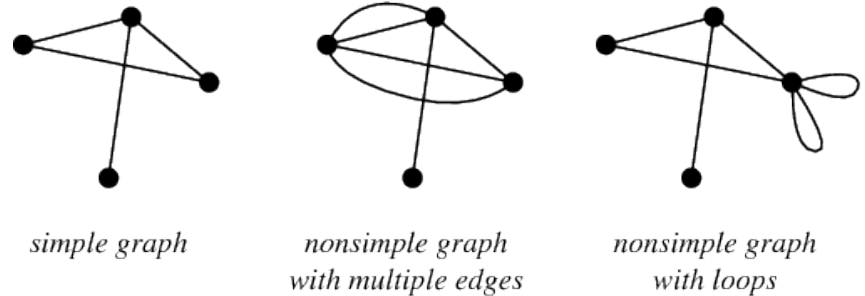
<http://spikedmath.com/382.html>

Big warning: Graph \neq Graph \neq Graph

Graph (plot)
(italiano: grafico)

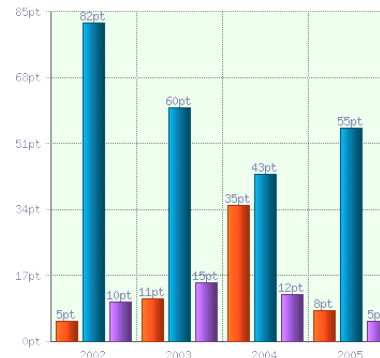


Graph (maths)
(italiano: grafo)



\neq

Graph (chart)
(italiano: grafico)



History

- ▶ The study of graphs is known as **graph theory**, and was first systematically investigated by D. König in the 1930s
- ▶ Euler's proof about the *walk across all seven bridges of Königsberg* (1736), now known as the *Königsberg bridge problem*, is a famous precursor to graph theory.
- ▶ In fact, the study of various sorts of paths in graphs has many applications in real-world problems.

Königsberg Bridge Problem

- ▶ Can the 7 bridges of the city of Königsberg over the river Pregel all be traversed in a single trip without doubling back, with the additional requirement that the trip ends in the same place it began?

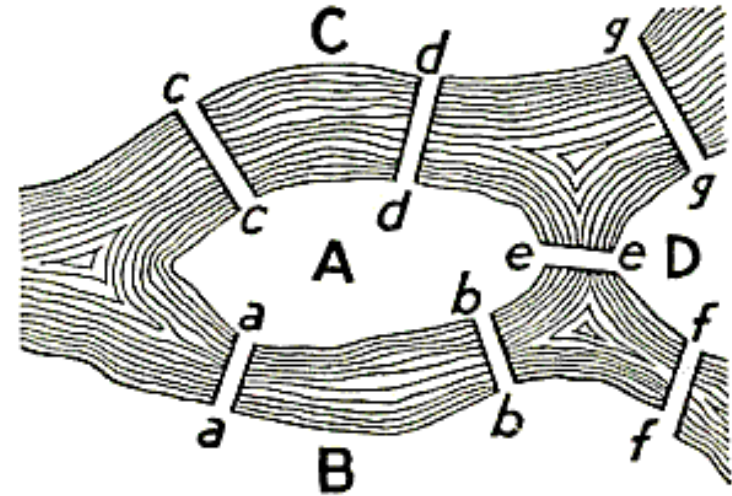
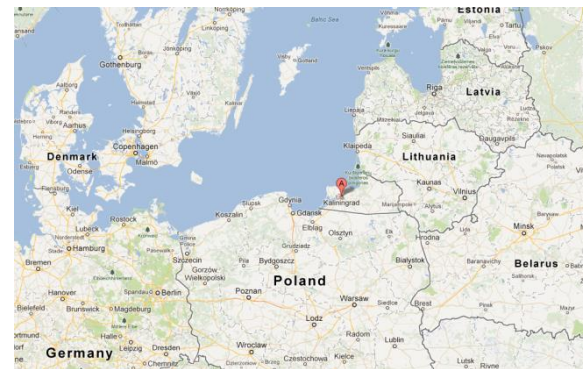


FIGURE 98. *Geographic Map:
The Königsberg Bridges.*



Today: Kaliningrad, Russia

Königsberg Bridge Problem

- ▶ Can the 7 bridges of the city of Königsberg over the river Pregel all be traversed in a single trip without crossing any bridge twice, and returning to the starting point?

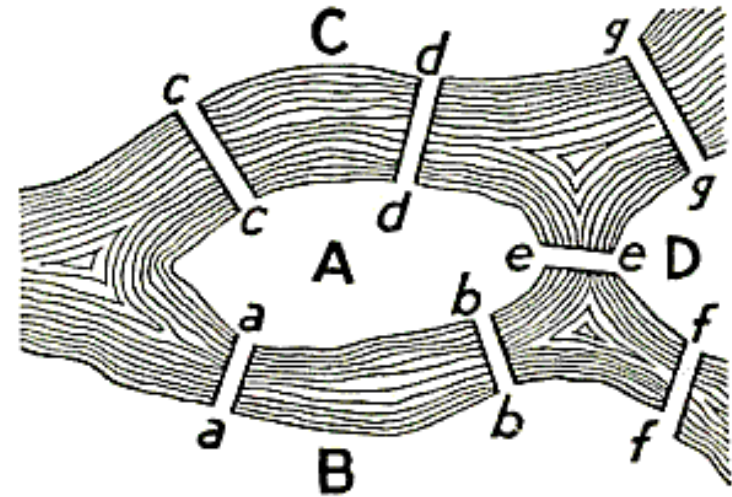
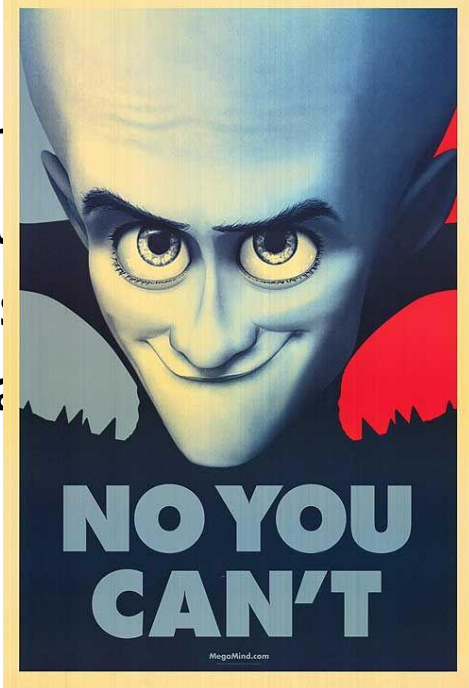
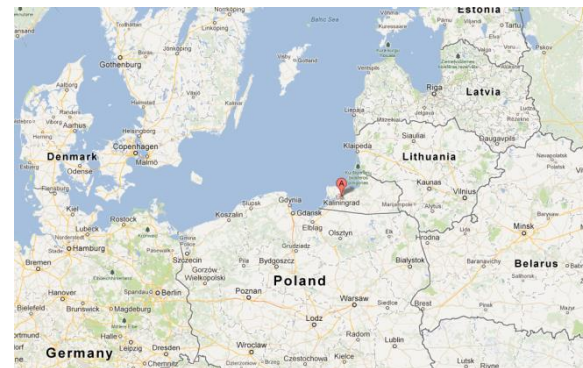
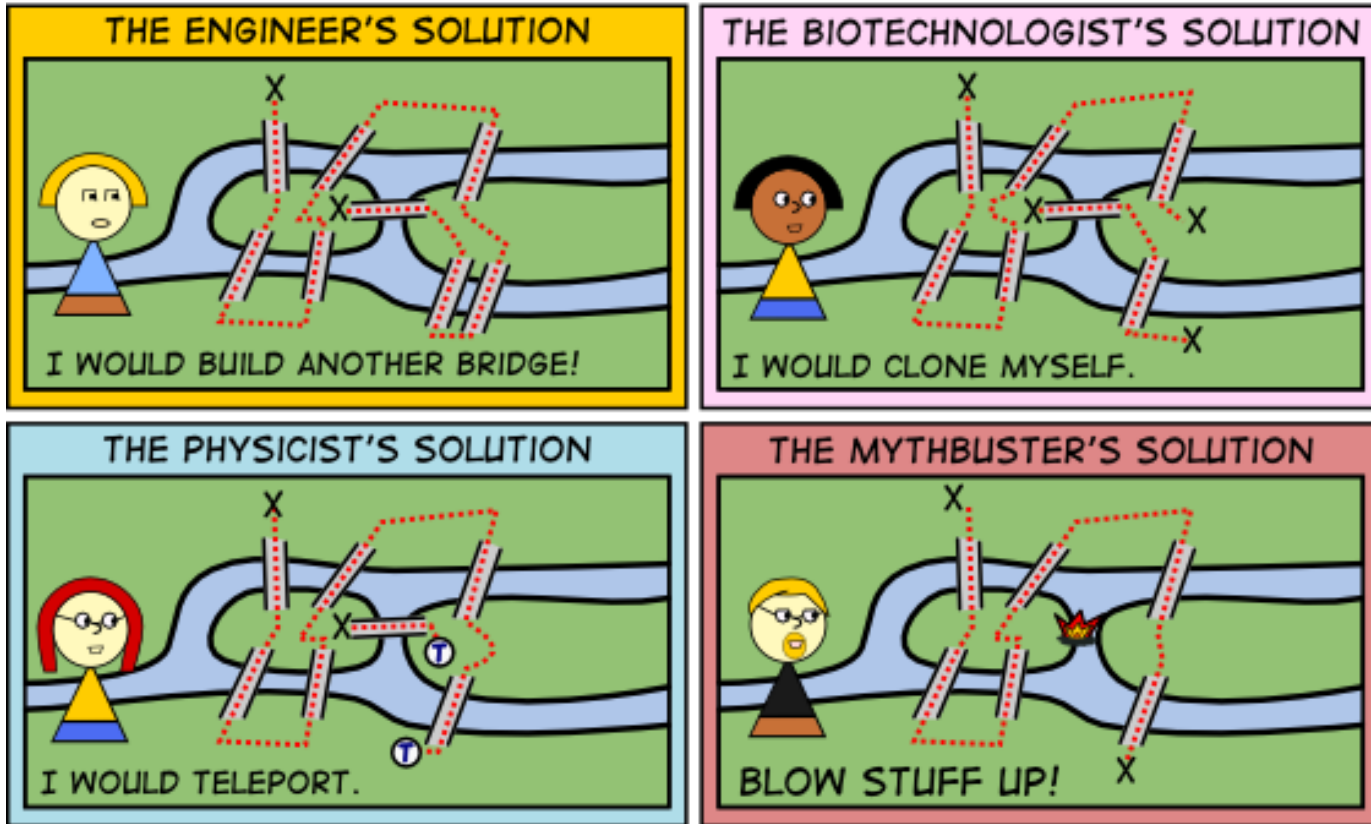


FIGURE 98. *Geographic Map:
The Königsberg Bridges.*



Today: Kaliningrad, Russia

Unless...

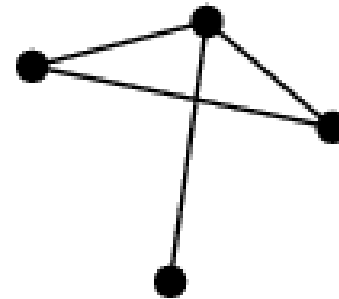


<http://spikedmath.com/541.html>

Types of graphs: edge cardinality

- ▶ **Simple graph:**

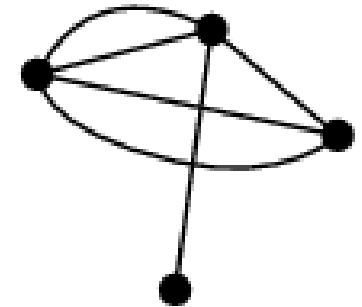
- ▶ At most one edge (i.e., either one edge or no edges) may connect any two vertices



simple graph

- ▶ **Multigraph:**

- ▶ Multiple edges are allowed between vertices



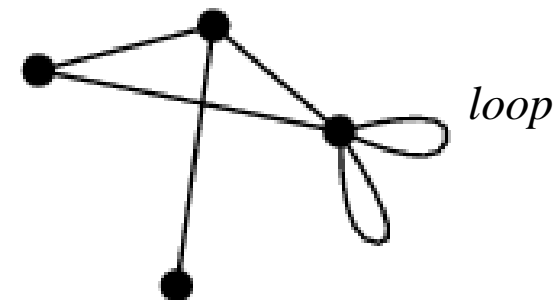
multigraph

- ▶ **Loops:**

- ▶ Edge between a vertex and itself

- ▶ **Pseudograph:**

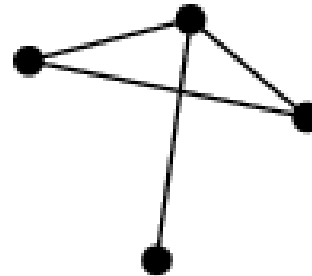
- ▶ Multigraph with loops



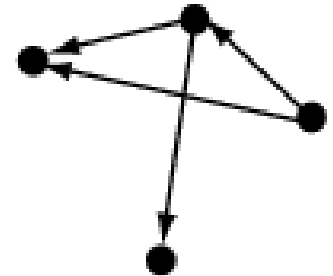
pseudograph

Types of graphs: edge direction

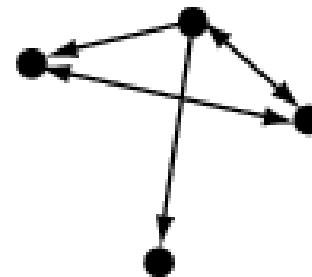
- ▶ Undirected
- ▶ Oriented
 - ▶ Edges have **one** direction (indicated by arrow)
- ▶ Directed
 - ▶ Edges may have **one or two** directions
- ▶ Network
 - ▶ Oriented graph with weighted edges



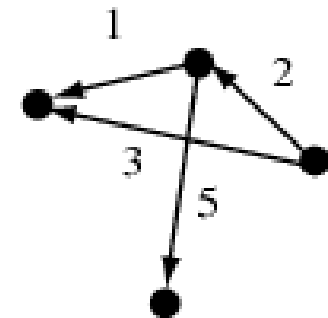
undirected graph



oriented graph



directed graph

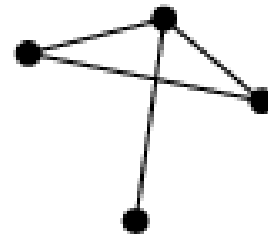


network

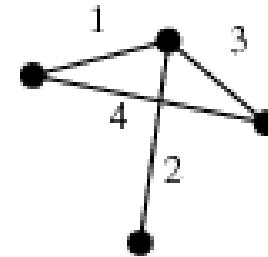
Types of graphs: labeling

▶ Labels

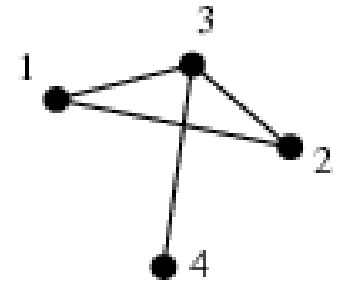
- ▶ None
- ▶ On Vertices
- ▶ On Edges



unlabeled graph



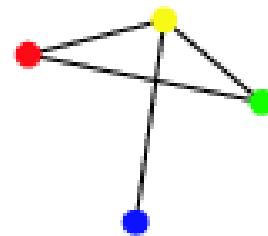
edge-labeled graph



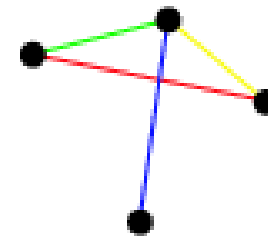
vertex-labeled graph

▶ Groups (=colors)

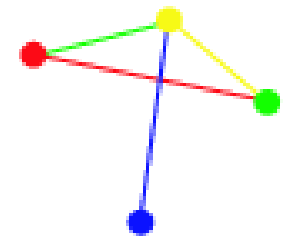
- ▶ Of Vertices
 - ▶ no edge connects two identically colored vertices
- ▶ Of Edges
 - ▶ adjacent edges must receive different colors
- ▶ Of both



vertex-colored graph



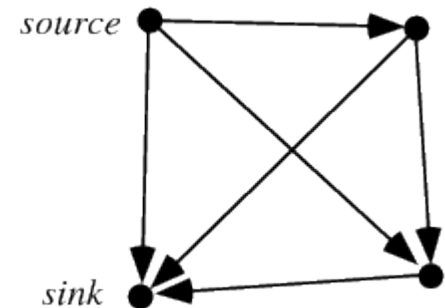
edge-colored graph



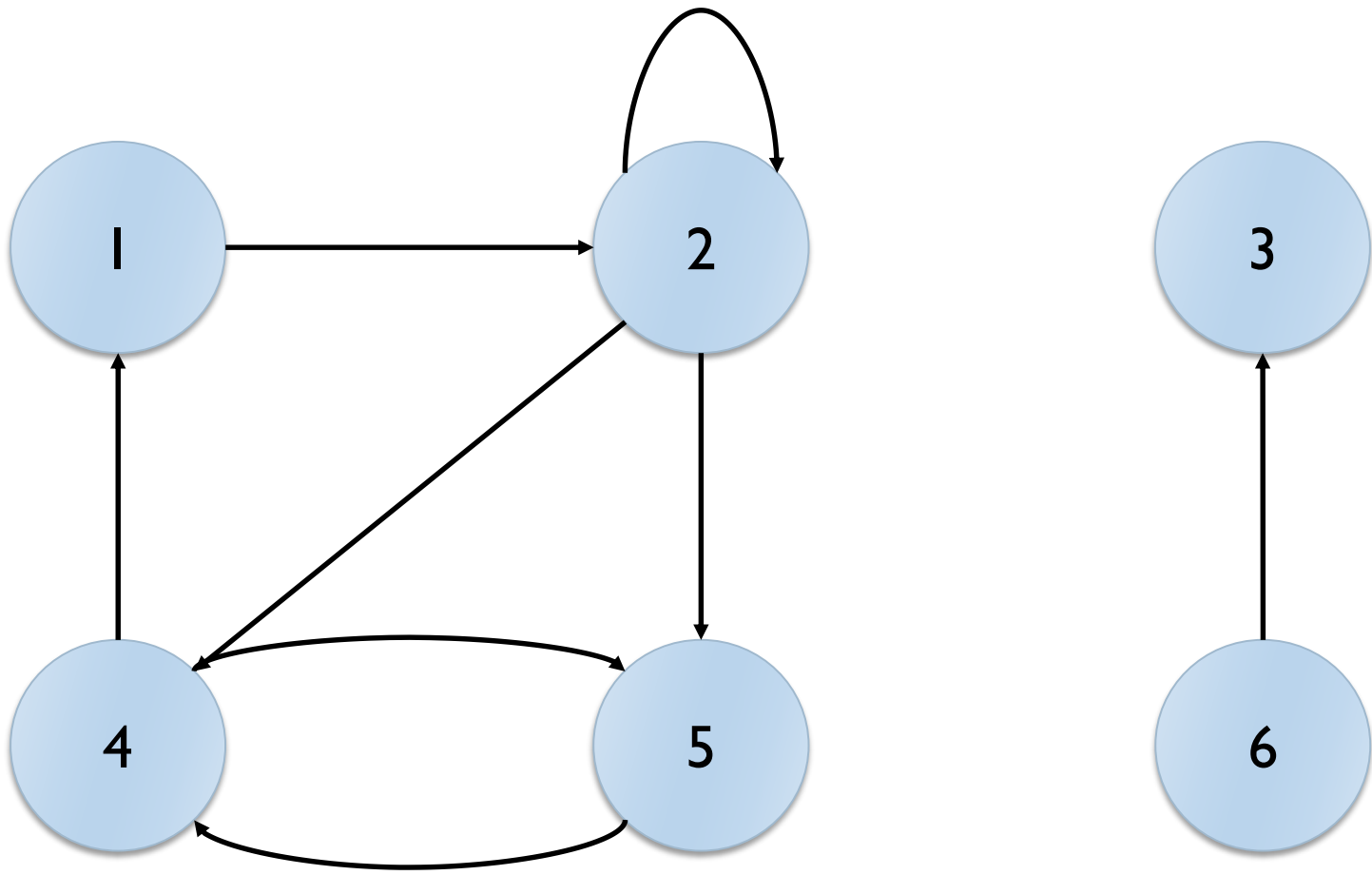
vertex- and edge-colored graph

Directed and Oriented graphs

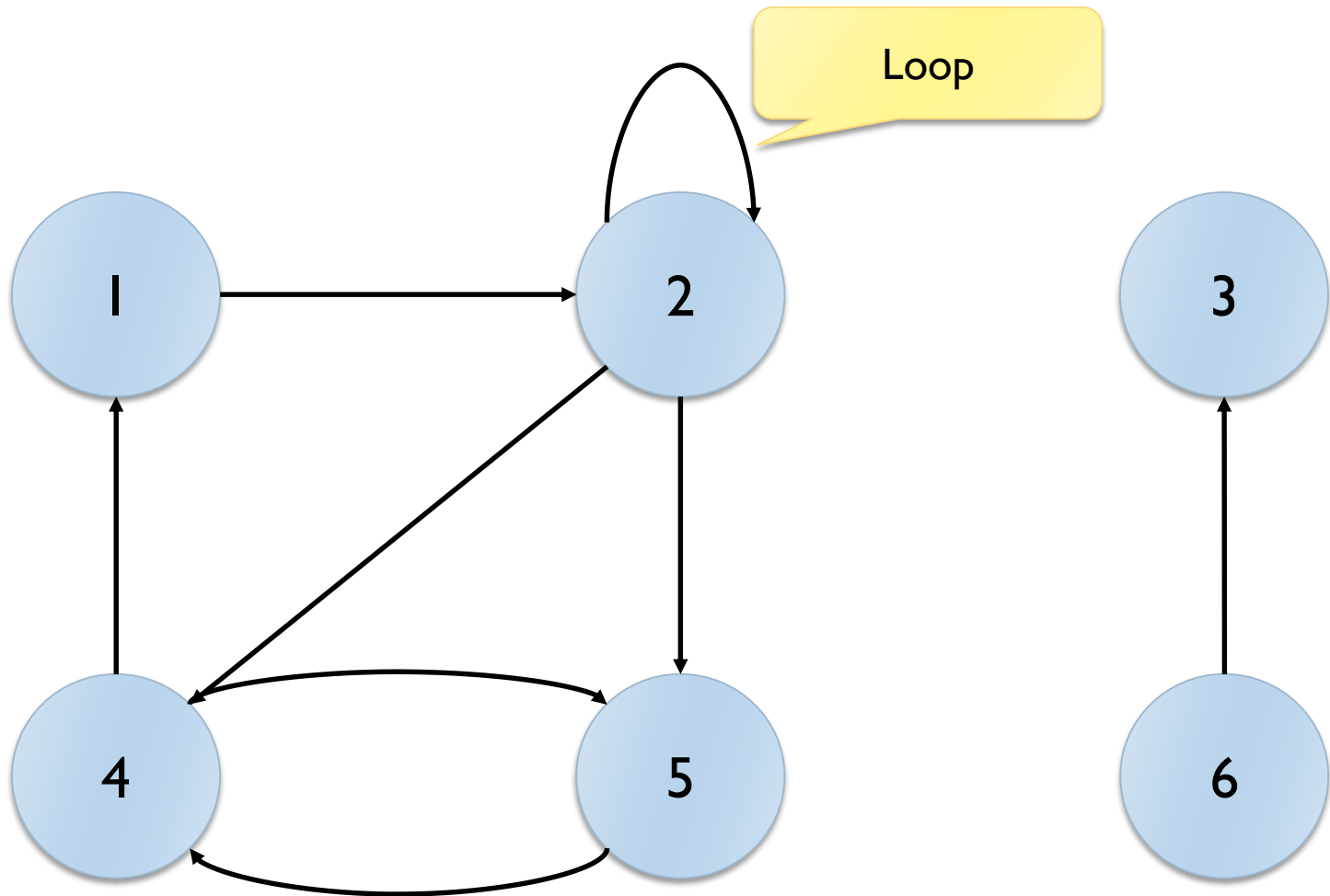
- ▶ A Directed Graph (*di-graph*) G is a pair (V,E) , where
 - ▶ V is a (finite) set of *vertices*
 - ▶ E is a (finite) set of *edges*, that identify a binary relationship over V
 - ▶ $E \subseteq V \times V$



Example



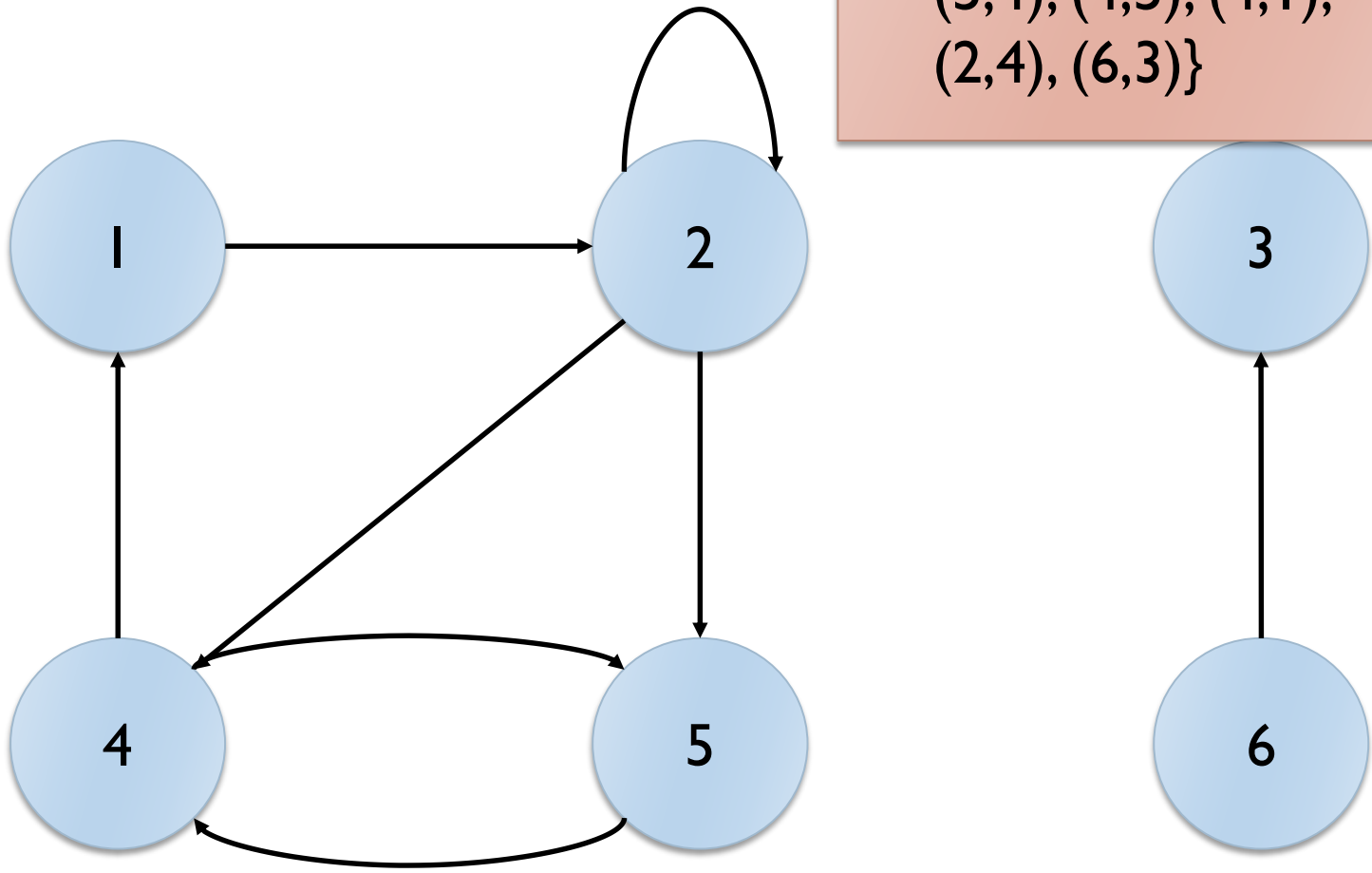
Example



Example

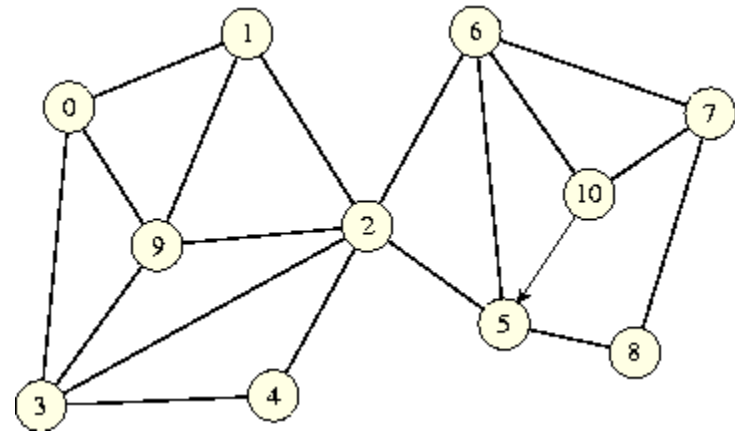
$$V = \{1, 2, 3, 4, 5, 6\}$$

$$E = \{(1, 2), (2, 2), (2, 5), (5, 4), (4, 5), (4, 1), (2, 4), (6, 3)\}$$



Undirected graph

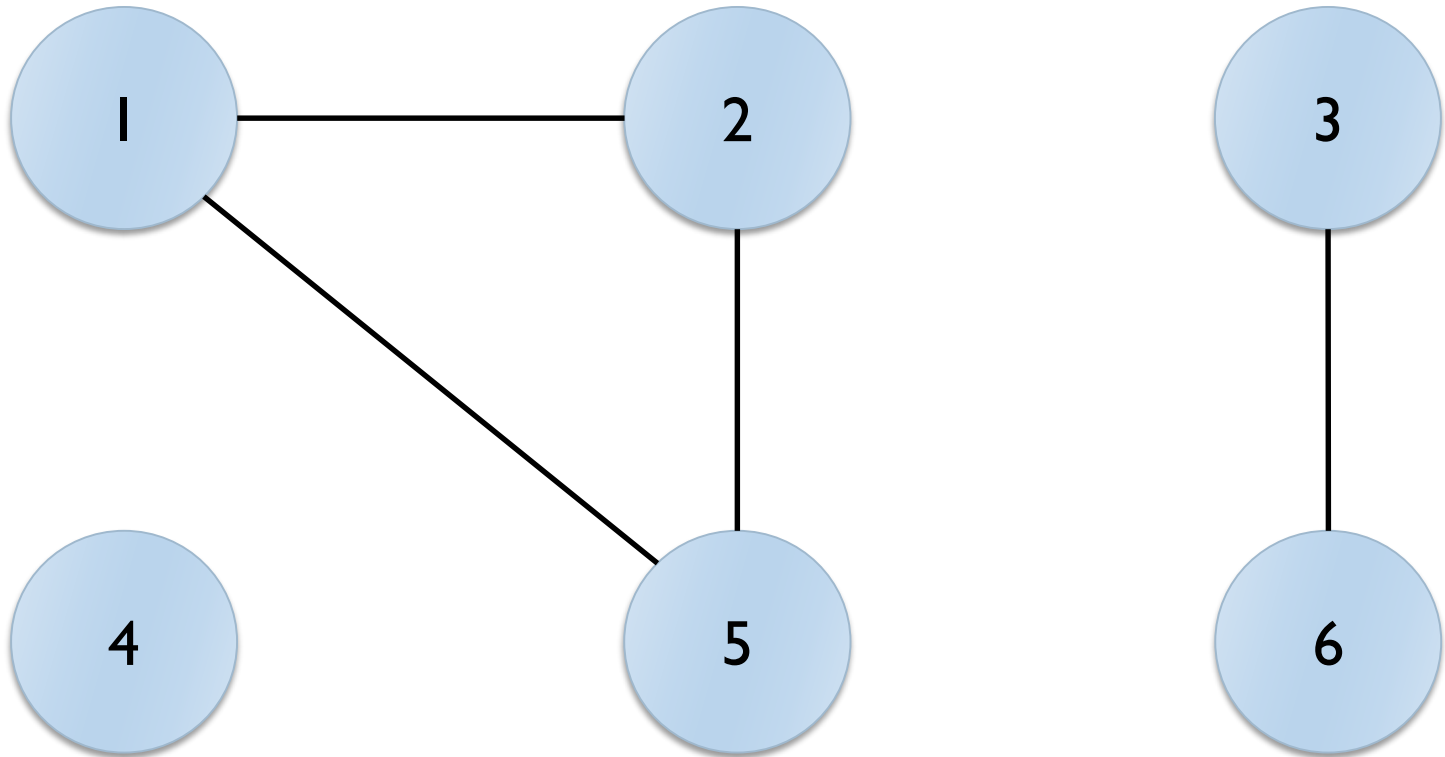
- ▶ Ad **Undirected** Graph is still represented as a couple $G=(V,E)$, but the set E is made of **non-ordered pairs** of vertices



Example

$V = \{1, 2, 3, 4, 5, 6\}$

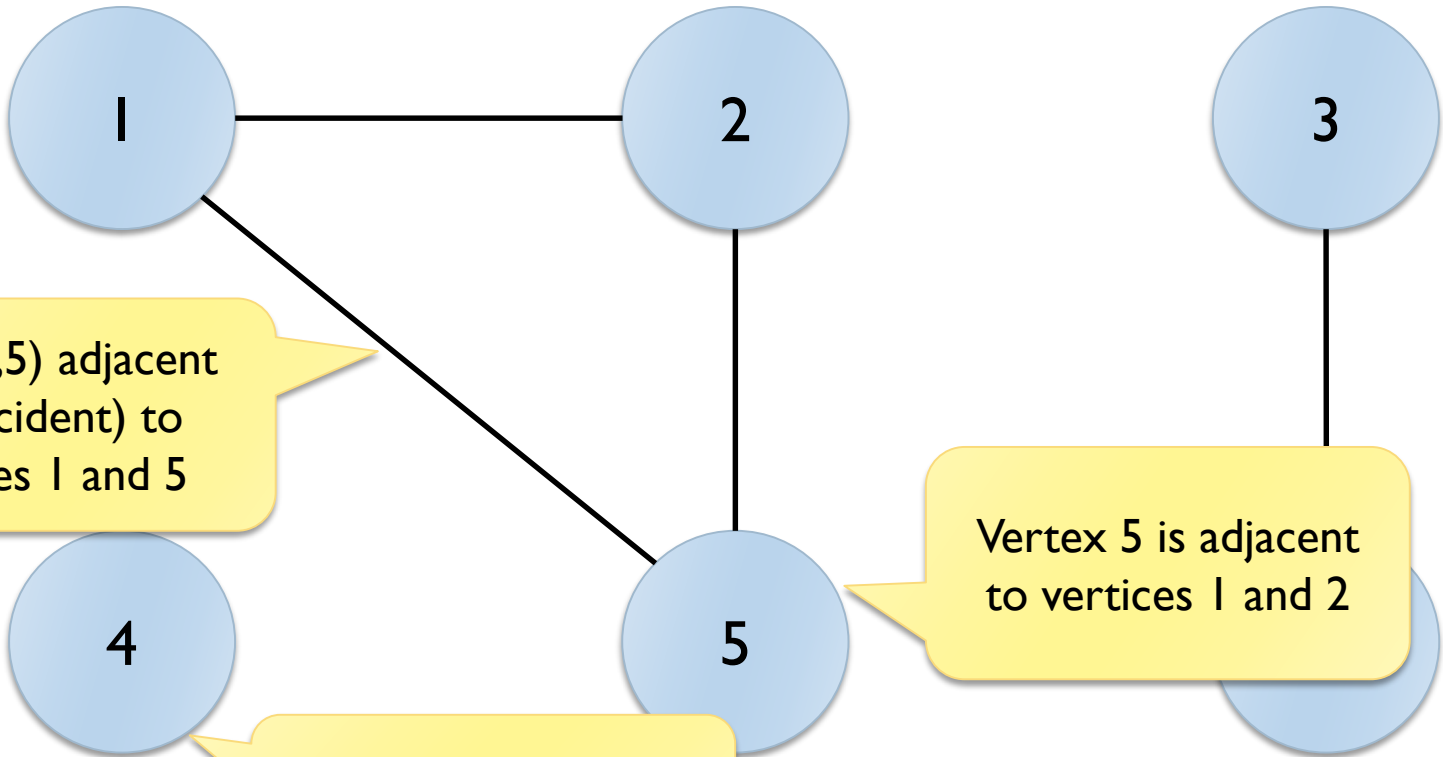
$E = \{\{1, 2\}, \{2, 5\}, \{5, 1\}, \{6, 3\}\}$



Example

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{\{1, 2\}, \{2, 5\}, \{5, 1\}, \{6, 3\}\}$



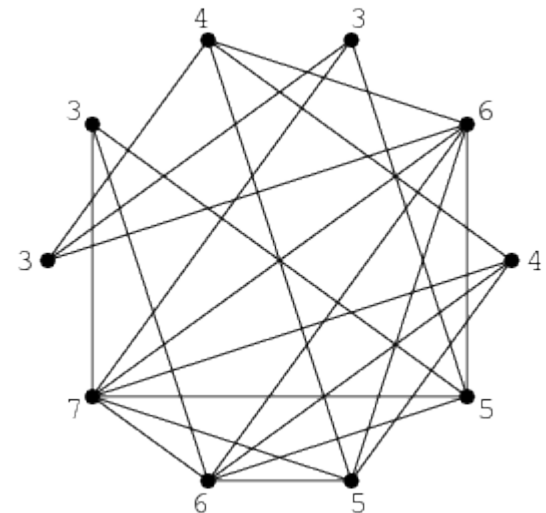
Edge (1,5) adjacent
(or incident) to
vertices 1 and 5

Vertex 5 is adjacent
to vertices 1 and 2

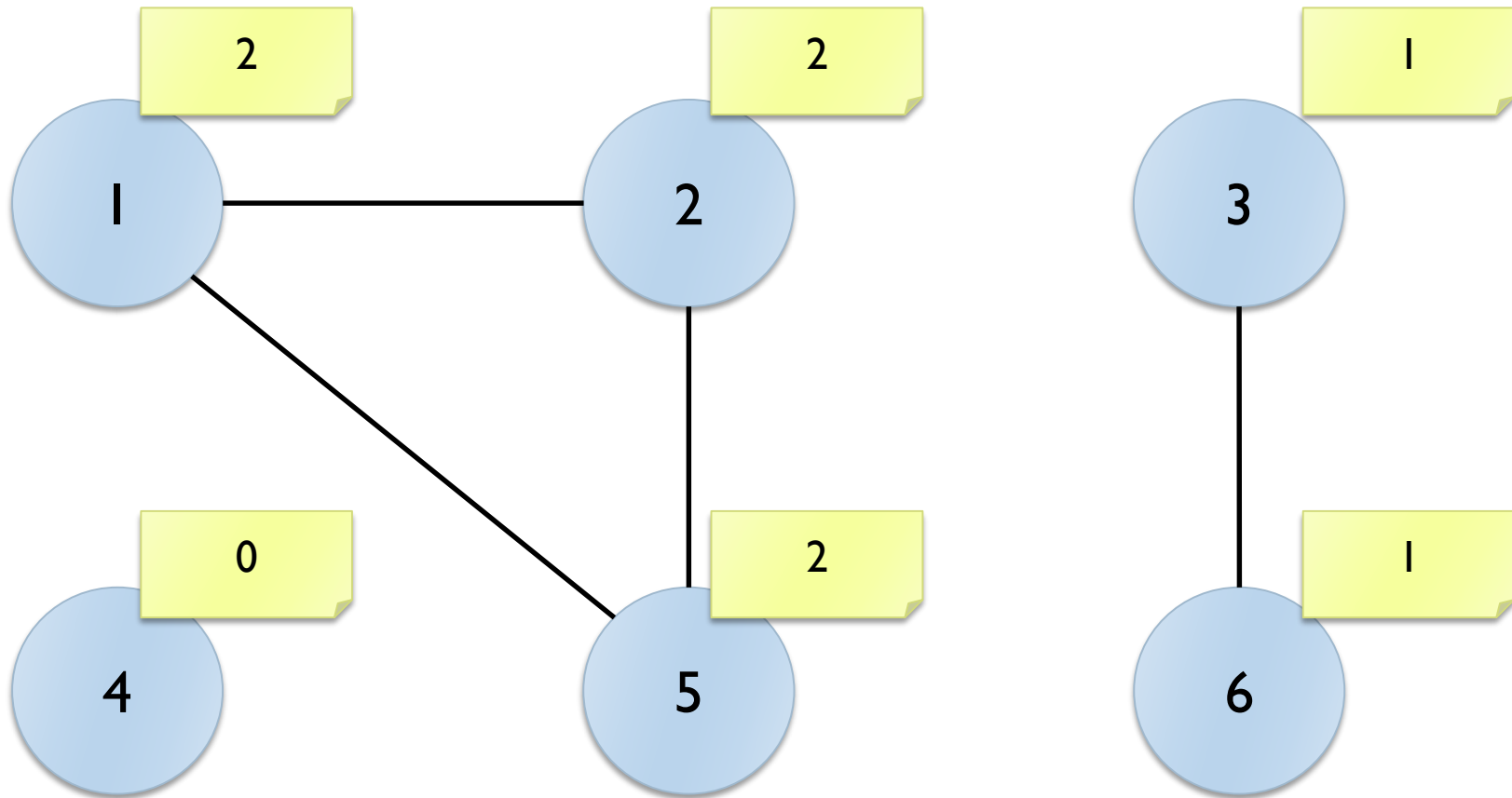
Vertex 4 is isolated

Degree

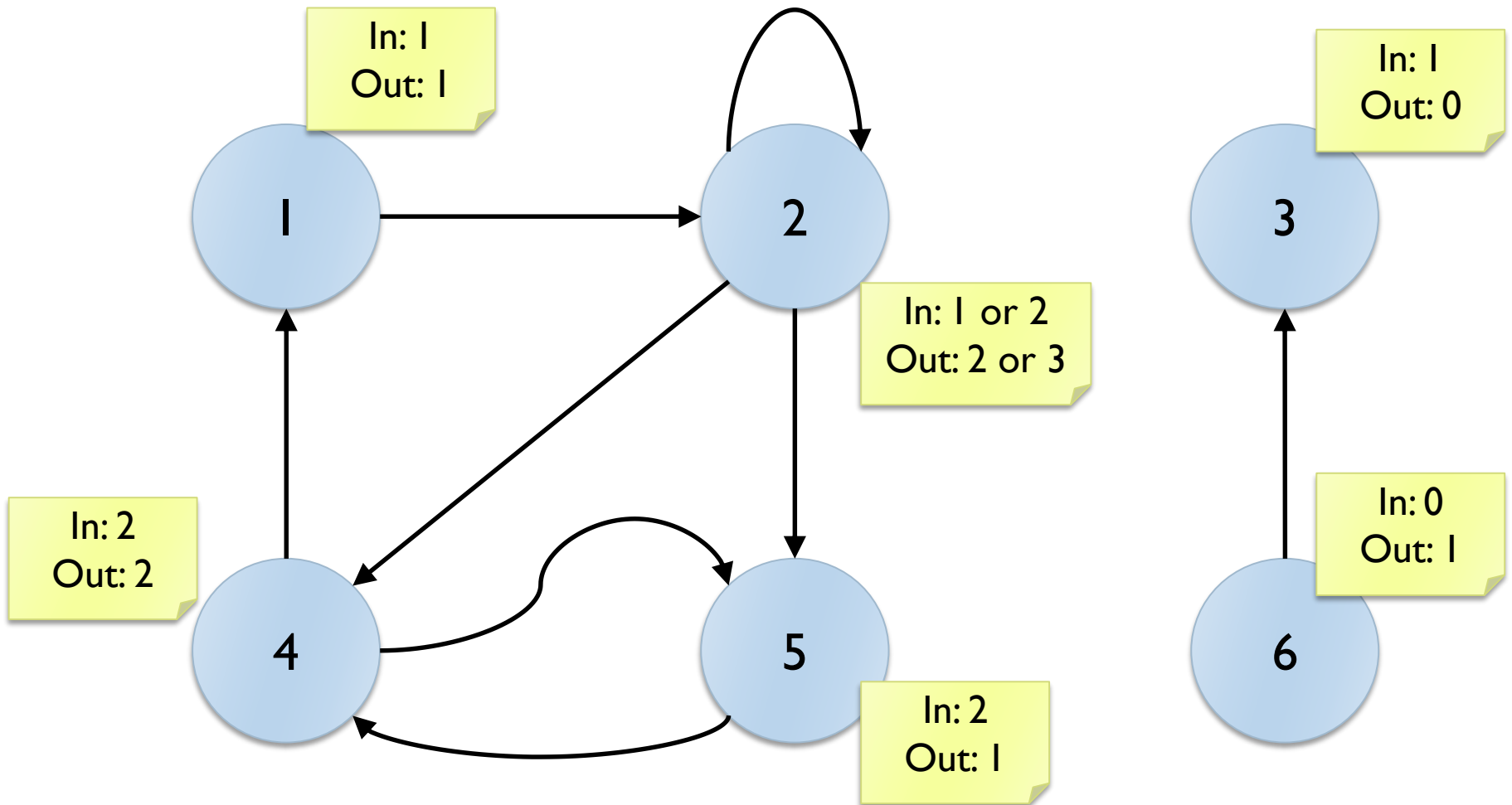
- ▶ In an *undirected* graph,
 - ▶ the **degree** of a vertex is the number of incident edges
- ▶ In a *directed* graph
 - ▶ The **in-degree** is the number of incoming edges
 - ▶ The **out-degree** is the number of departing edges
 - ▶ The **degree** is the sum of in-degree and out-degree
- ▶ A vertex with degree 0 is **isolated**



Degree



Degree

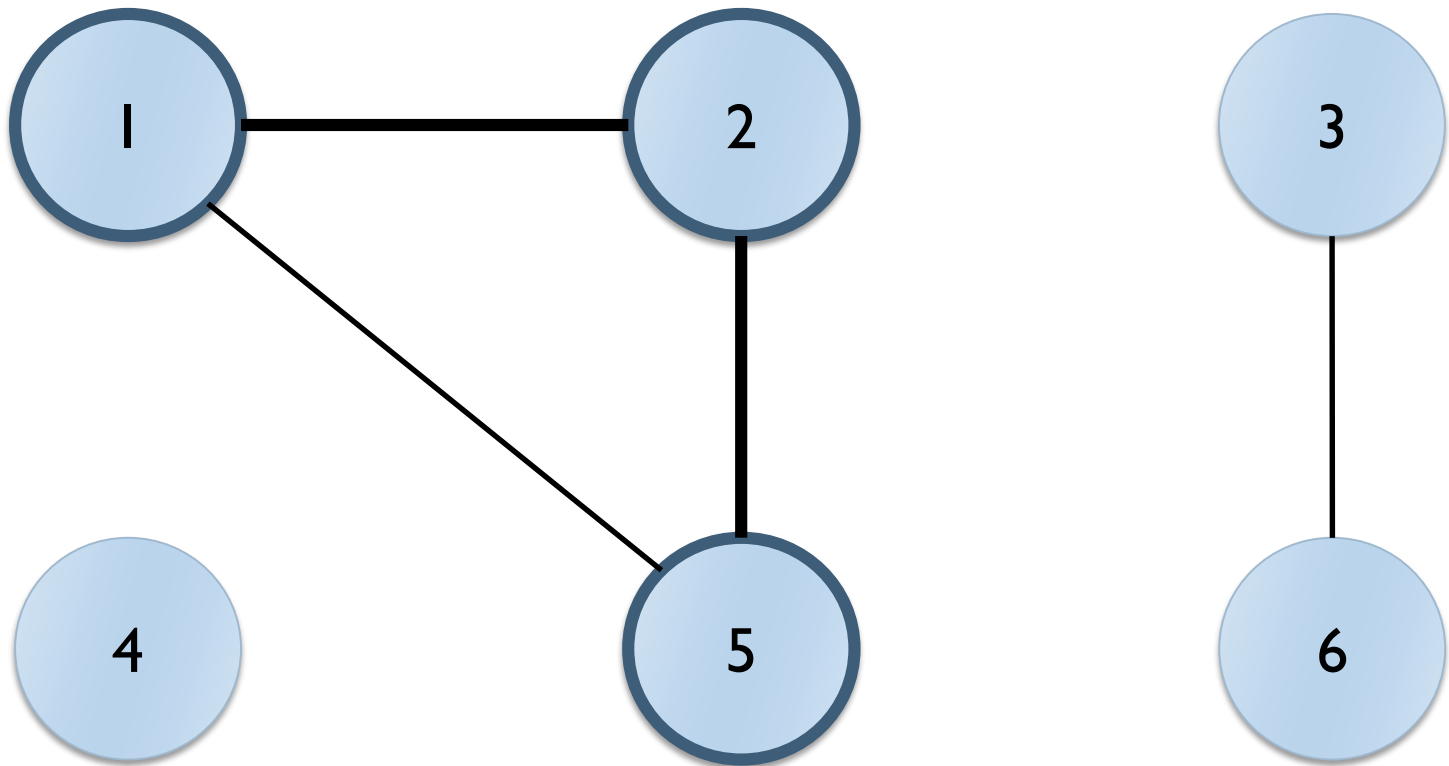


Paths

- ▶ A **path** on a graph $G=(V,E)$ also called a trail, is a sequence $\{v_1, v_2, \dots, v_n\}$ such that:
 - ▶ v_1, \dots, v_n are vertices: $v_i \in V$
 - ▶ $(v_1, v_2), (v_2, v_3), \dots, (v_{n-1}, v_n)$ are graph edges: $(v_{i-1}, v_i) \in E$
 - ▶ v_i are distinct (for “simple” paths).
- ▶ The **length** of a path is the number of edges $(n-1)$
- ▶ If there exist a path between v_A and v_B we say that v_B is **reachable** from v_A

Example

Path = { 1, 2, 5 }
Length = 2

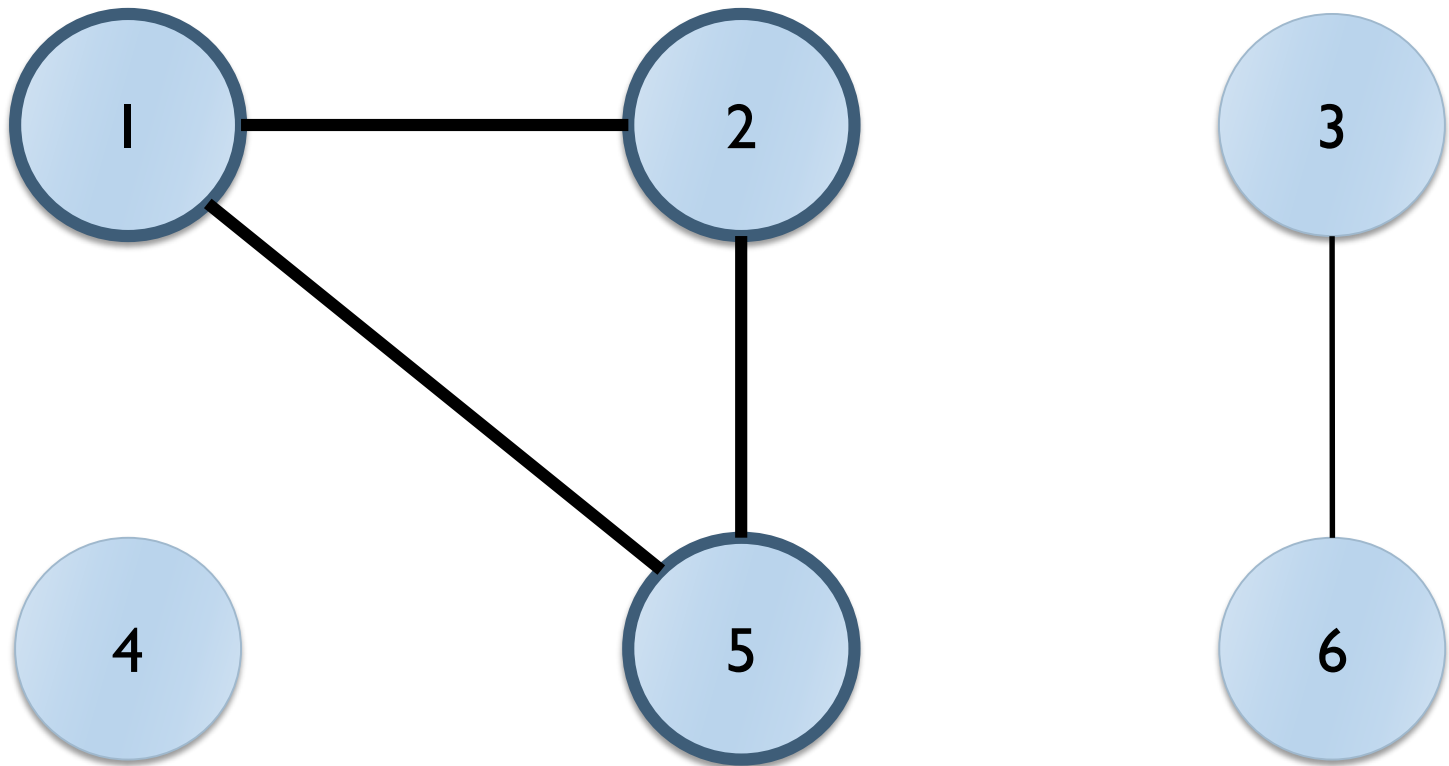


Cycles

- ▶ A cycle is a path where $v_1 = v_n$
- ▶ A graph with no cycles is said acyclic

Example

Path = { 1, 2, 5, 1 }
Length = 3



Reachability (Undirected)

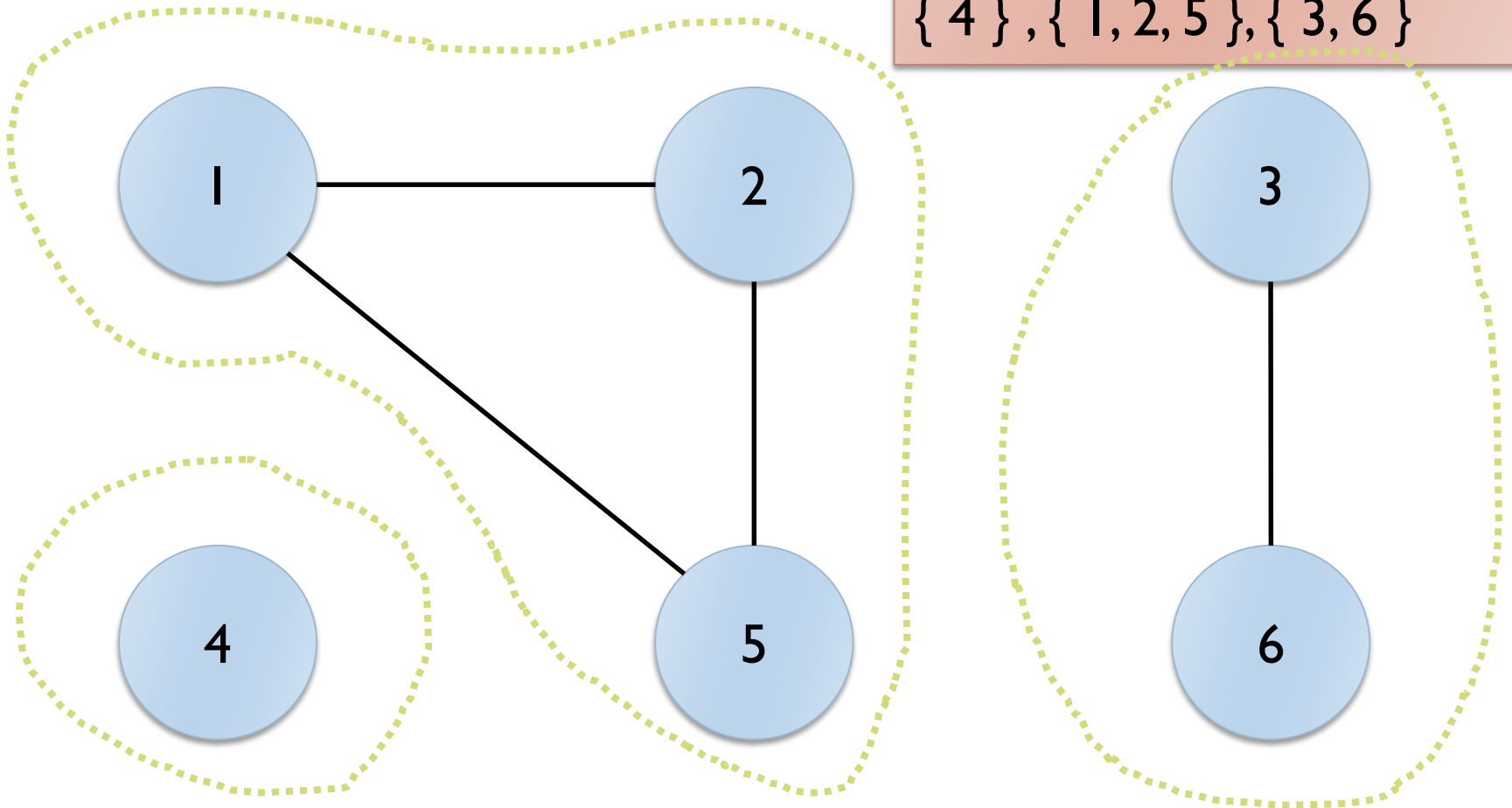
- ▶ An undirected graph is **connected** if, for every couple of vertices, there is a path connecting them
- ▶ The connected sub-graph of maximum size are called **connected components**
- ▶ A connected graph has exactly one connected component

Connected components

The graph is **not** connected.

Connected components = 3

$\{ 4 \}, \{ 1, 2, 5 \}, \{ 3, 6 \}$

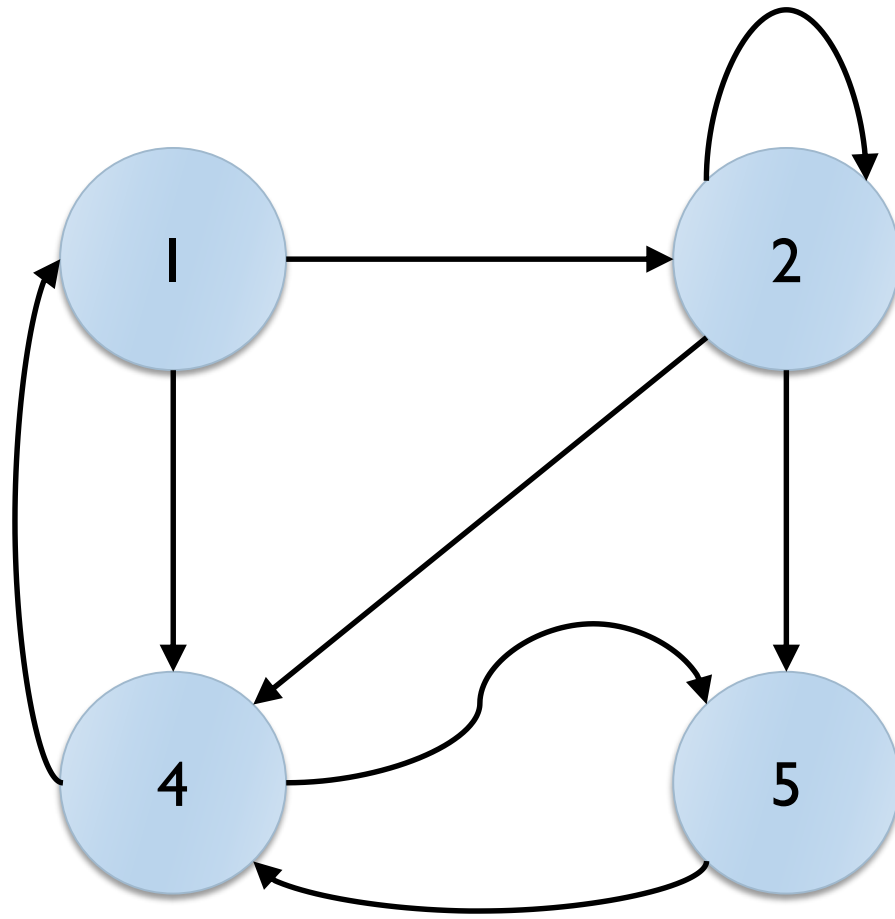


Reachability (Directed)

- ▶ A directed graph is **strongly connected** if, for every ordered pair of vertices (v, v') , there exists at least one path connecting v to v'

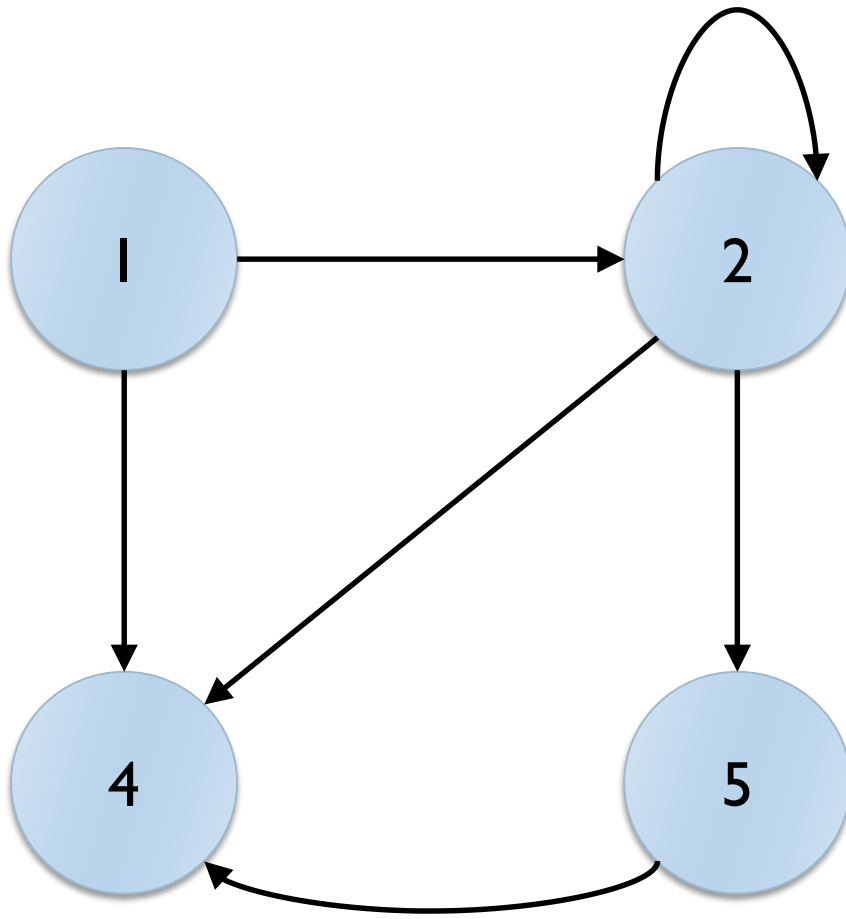
Example

The graph is **strongly connected**



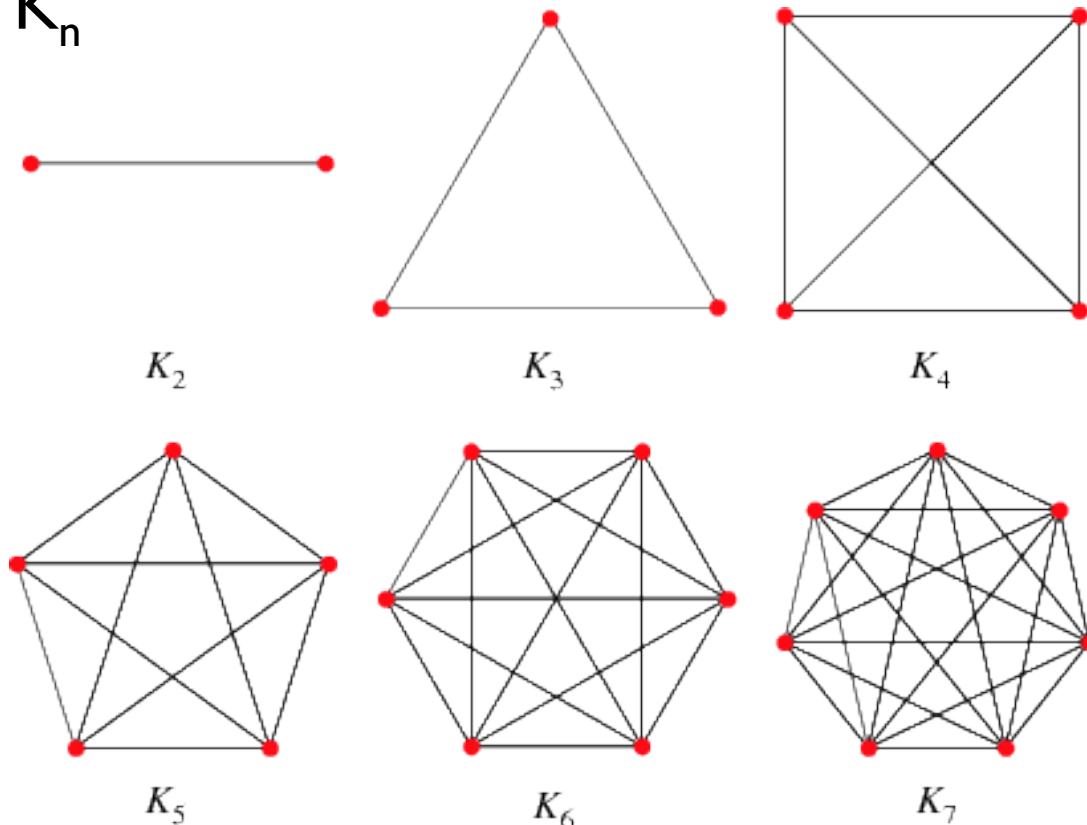
Example

The graph is **not** strongly connected



Complete graph

- ▶ A graph is complete if, for every pair of vertices, there is an edge connecting them (they are adjacent)
- ▶ Symbol: K_n



Complete graph: edges

- ▶ In a **complete** graph with n vertices, the number of **edges** is
 - ▶ $n(n-1)$, if the graph is directed
 - ▶ $n(n-1)/2$, if the graph is undirected
 - ▶ If self-loops are allowed, then
 - ▶ n^2 for directed graphs
 - ▶ $n(n-1)$ for undirected graphs

Density

- ▶ The density of a graph $G=(V,E)$ is the ratio of the number of edges to the total number of possible edges

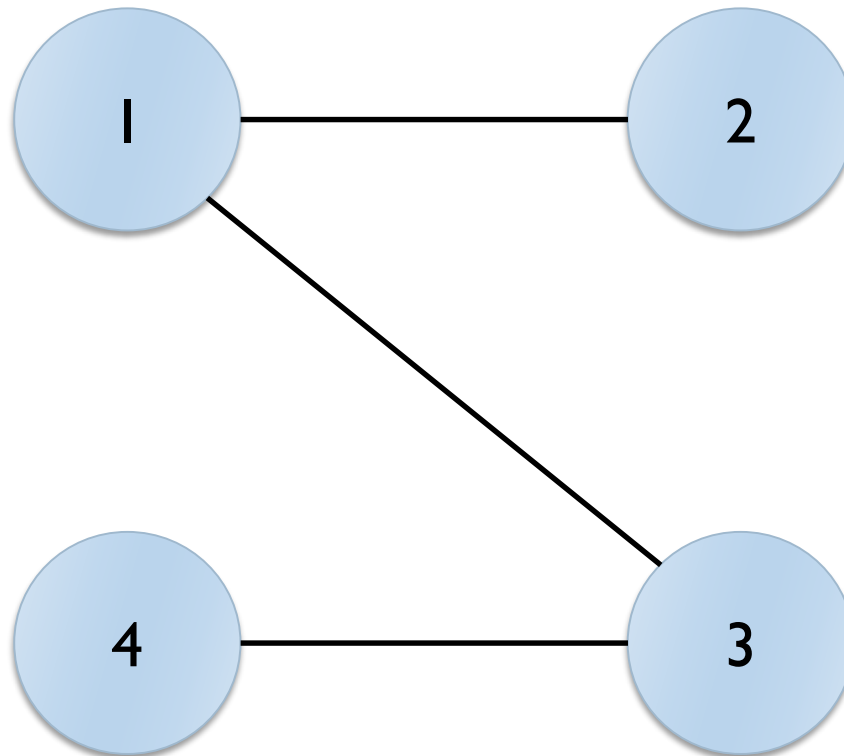
$$d = \frac{|E(G)|}{|E(K_{|V(G)|})|}$$

Esempio

Density = 0.5

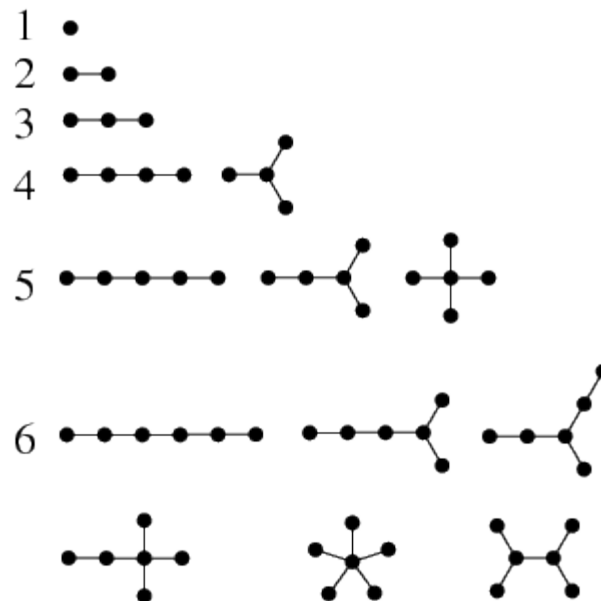
Existing: 3 edges

Total: 6 possible edges



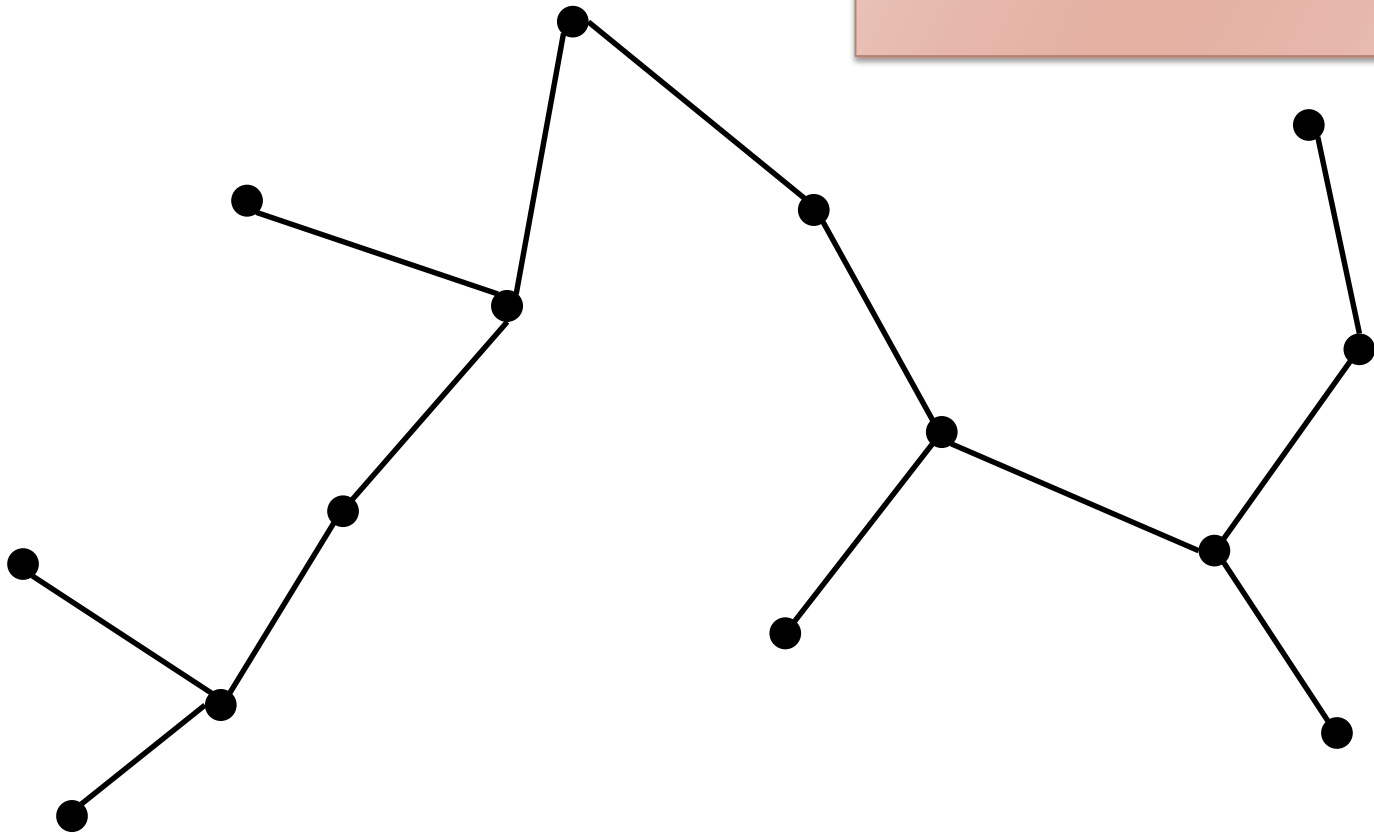
Trees and Forests

- ▶ An undirected acyclic graph is called **forest**
- ▶ An undirected acyclic connected graph is called **tree**



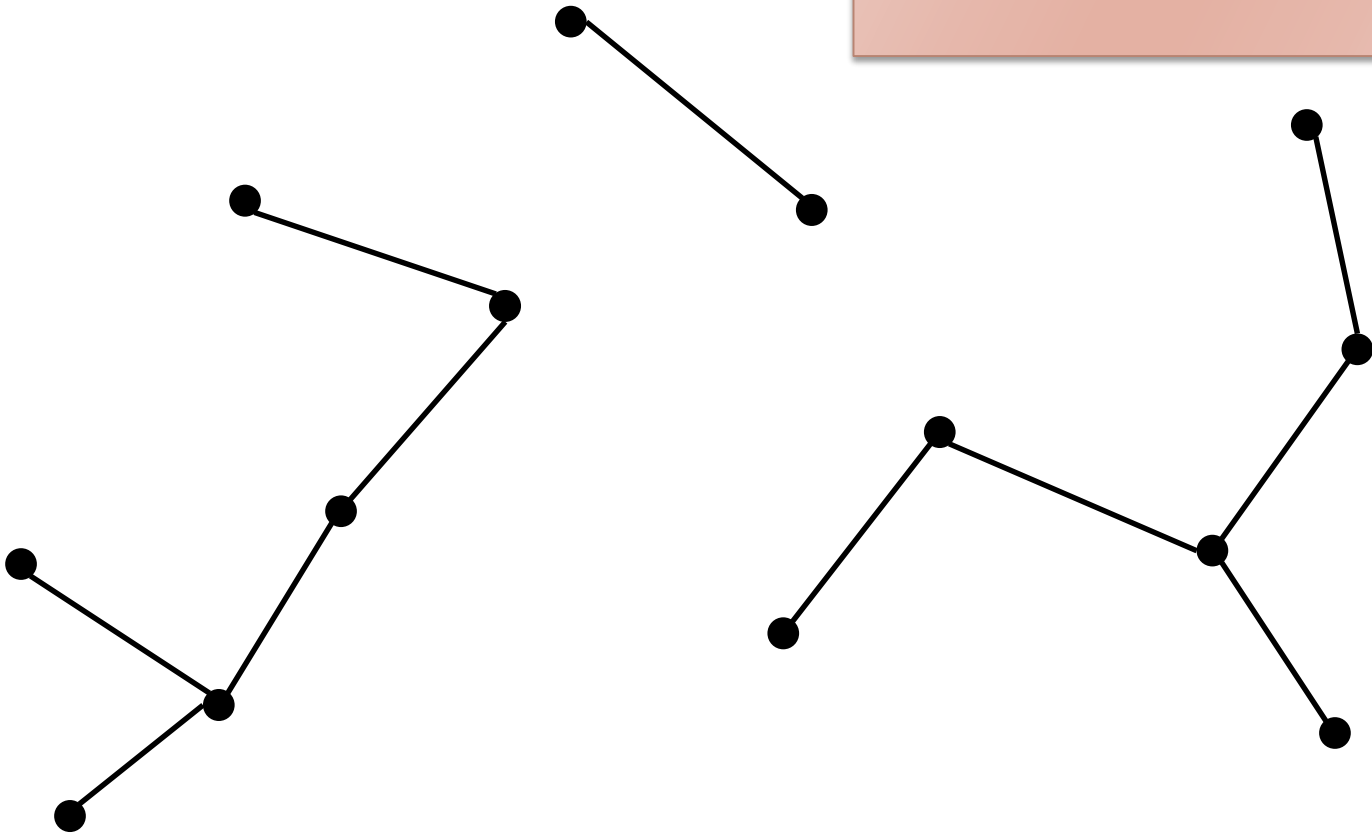
Example

Tree



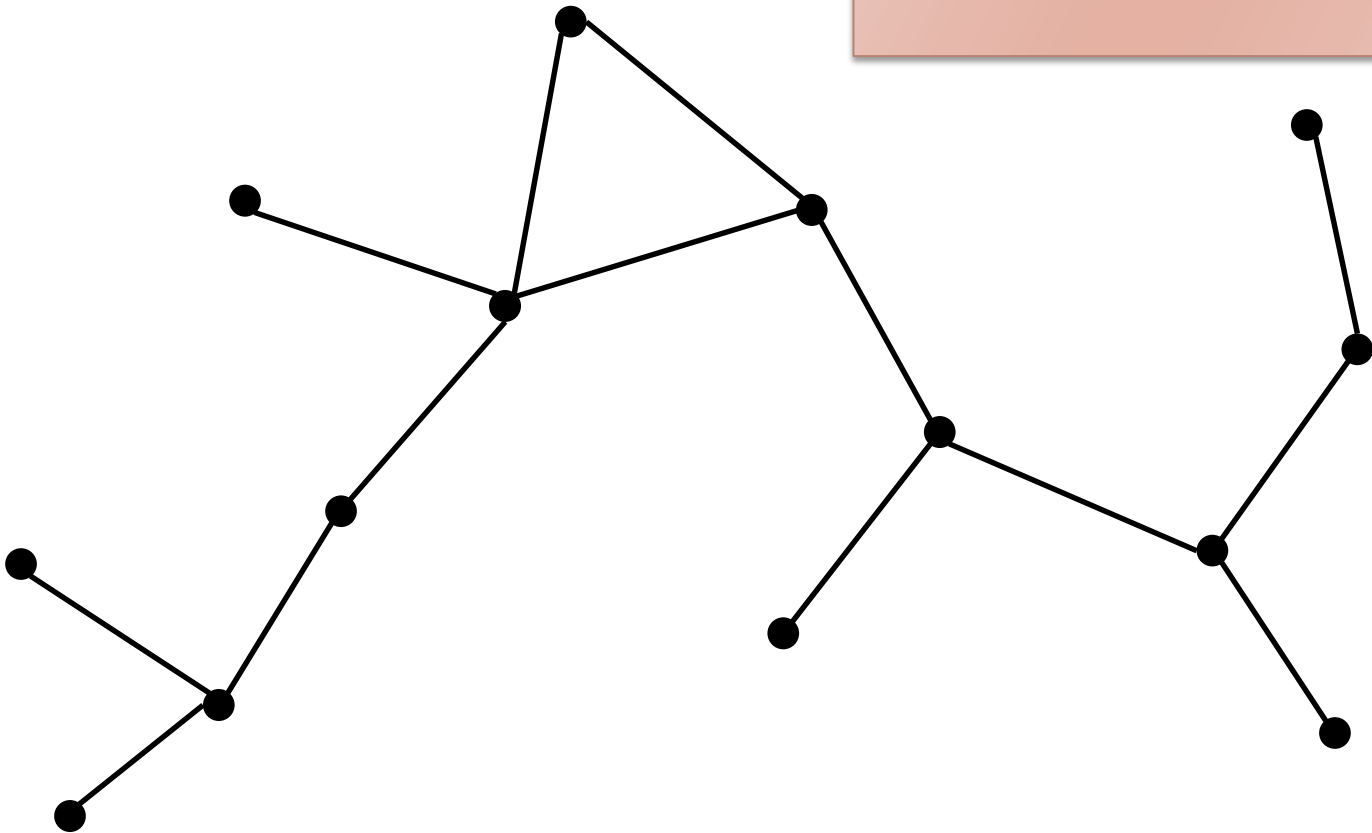
Example

Forest



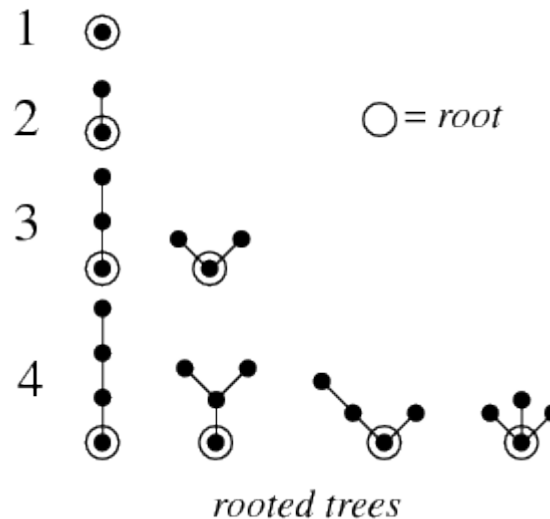
Example

This is not a tree nor a forest
(it contains a cycle)



Rooted trees

- ▶ In a tree, a special node may be singled out
- ▶ This node is called the “**root**” of the tree
- ▶ Any node of a tree can be the root

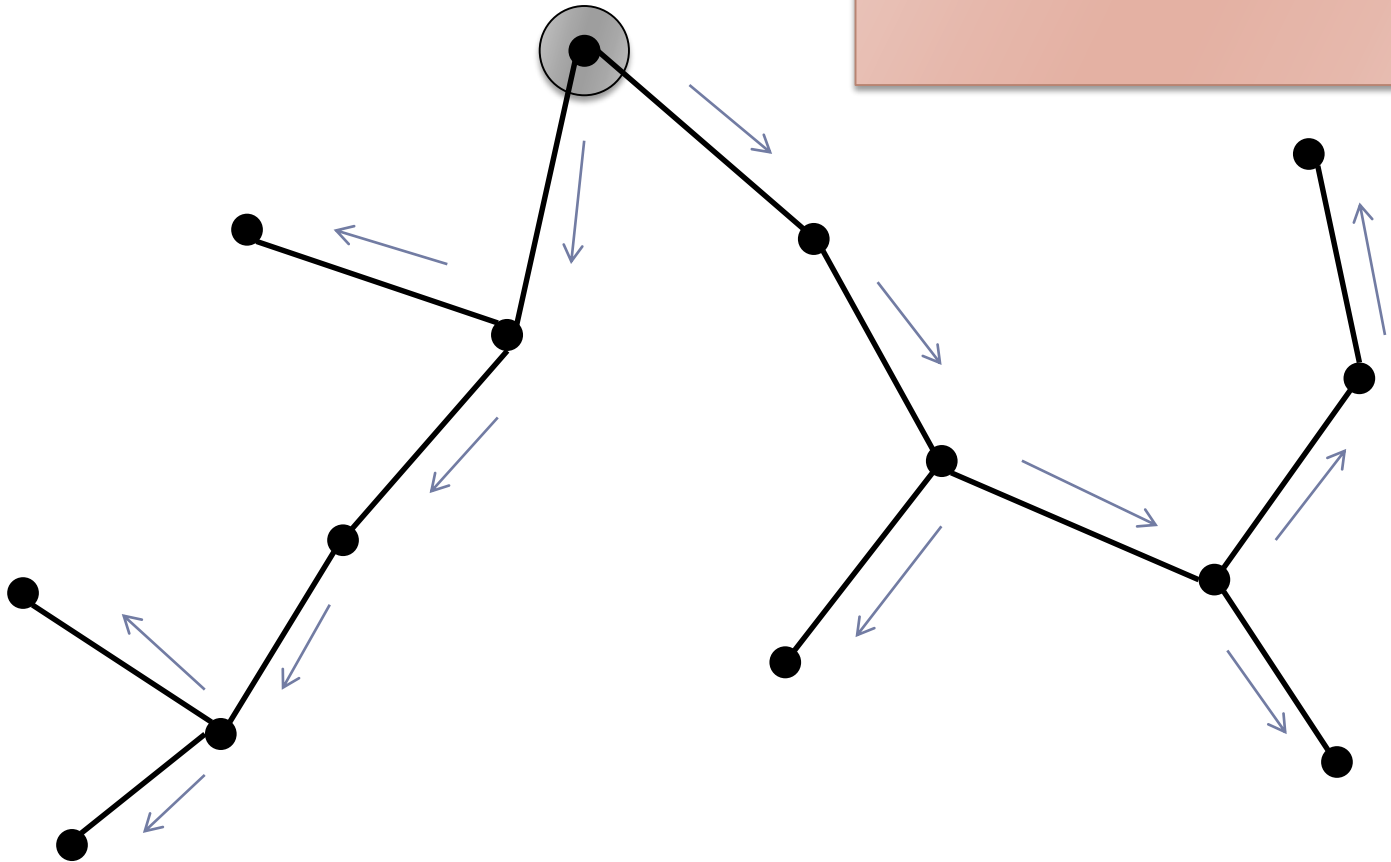


Tree (implicit) ordering

- ▶ The root node of a tree **induces an ordering** of the nodes
- ▶ The root is the “ancestor” of all other nodes/vertices
 - ▶ “children” are “away from the root”
 - ▶ “parents” are “towards the root”
- ▶ The root is the only node without parents
- ▶ All other nodes have exactly one parent
- ▶ The furthestmost (children-of-children-of-children...) nodes are “leaves”

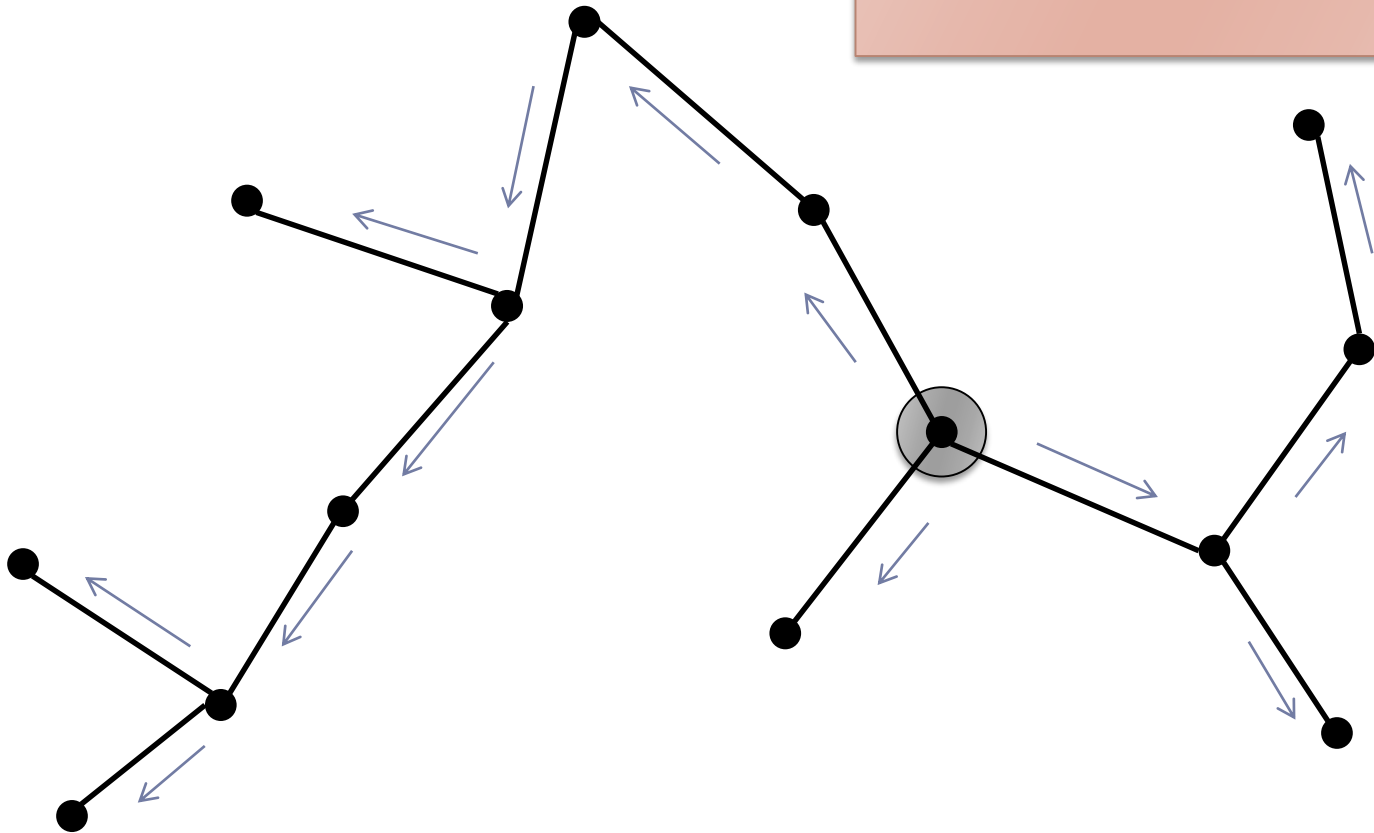
Example

Rooted Tree



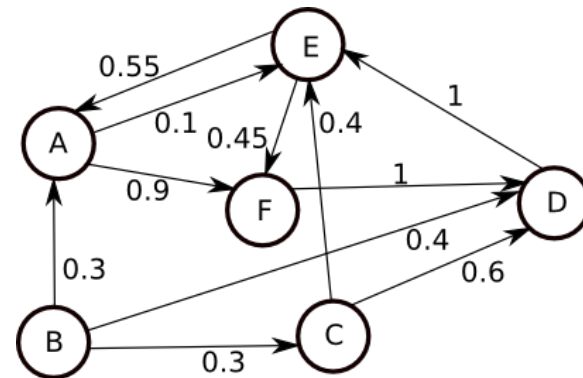
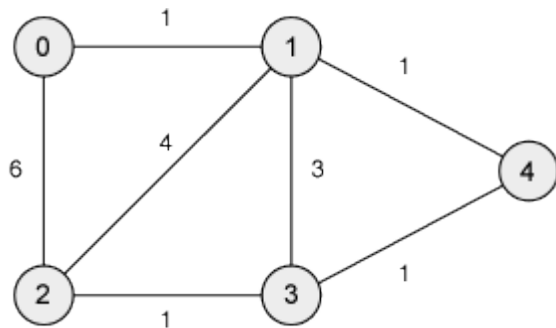
Example

Rooted Tree



Weighted graphs

- ▶ A weighted graph is a graph in which each branch (edge) is given a numerical weight.
- ▶ A weighted graph is therefore a special type of labeled graph in which the labels are numbers (which are usually taken to be positive).





Applications

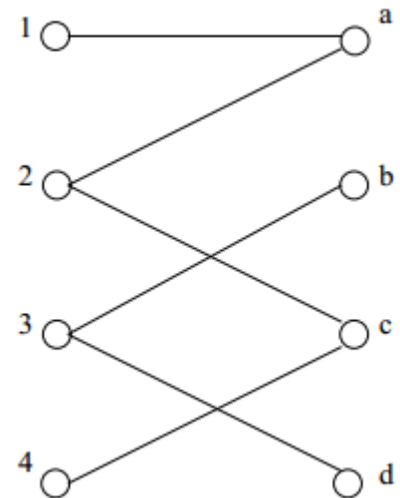
Introduction to Graphs

Graph applications

- ▶ **Graphs are everywhere**
 - ▶ Facebook friends (and posts, and 'likes')
 - ▶ Football tournaments (complete subgraphs + binary tree)
 - ▶ Google search index (V =page, E =link, w =pagerank)
 - ▶ Web analytics (site structure, visitor paths)
 - ▶ Car navigation (GPS)
 - ▶ Market Matching

Market matching

- ▶ $H = \text{Houses } (1, 2, 3, 4)$
- ▶ $B = \text{Buyers } (a, b, c, d)$
- ▶ $V = H \cup B$
- ▶ Edges: $(h, b) \in E$ if b would like to buy h
- ▶ Problem: can all houses be sold and all buyers be satisfied?
- ▶ Variant: if the graph is weighted with a purchase offer, what is the most convenient solution?
- ▶ Variant: consider a 'penalty' for unsold items

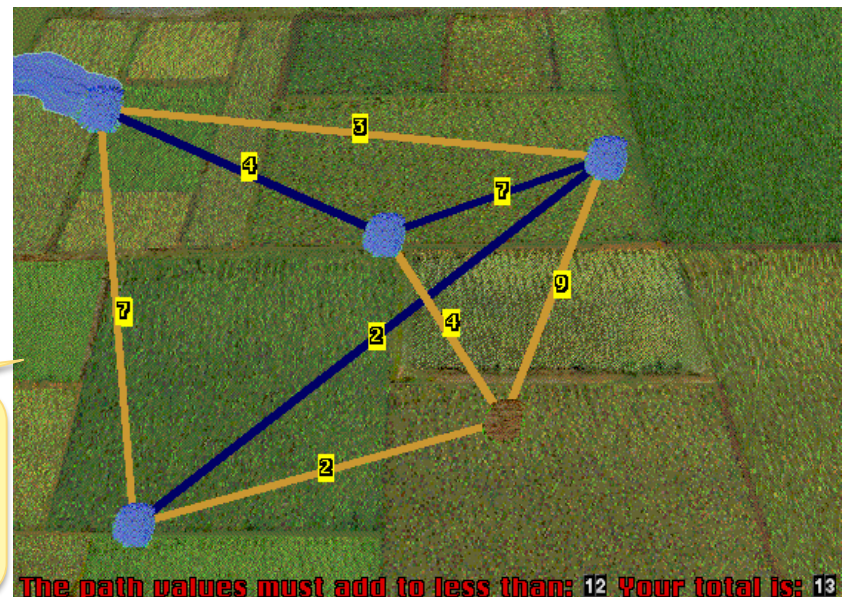


This graph is called
“bipartite”:
 $H \cap B = \emptyset$

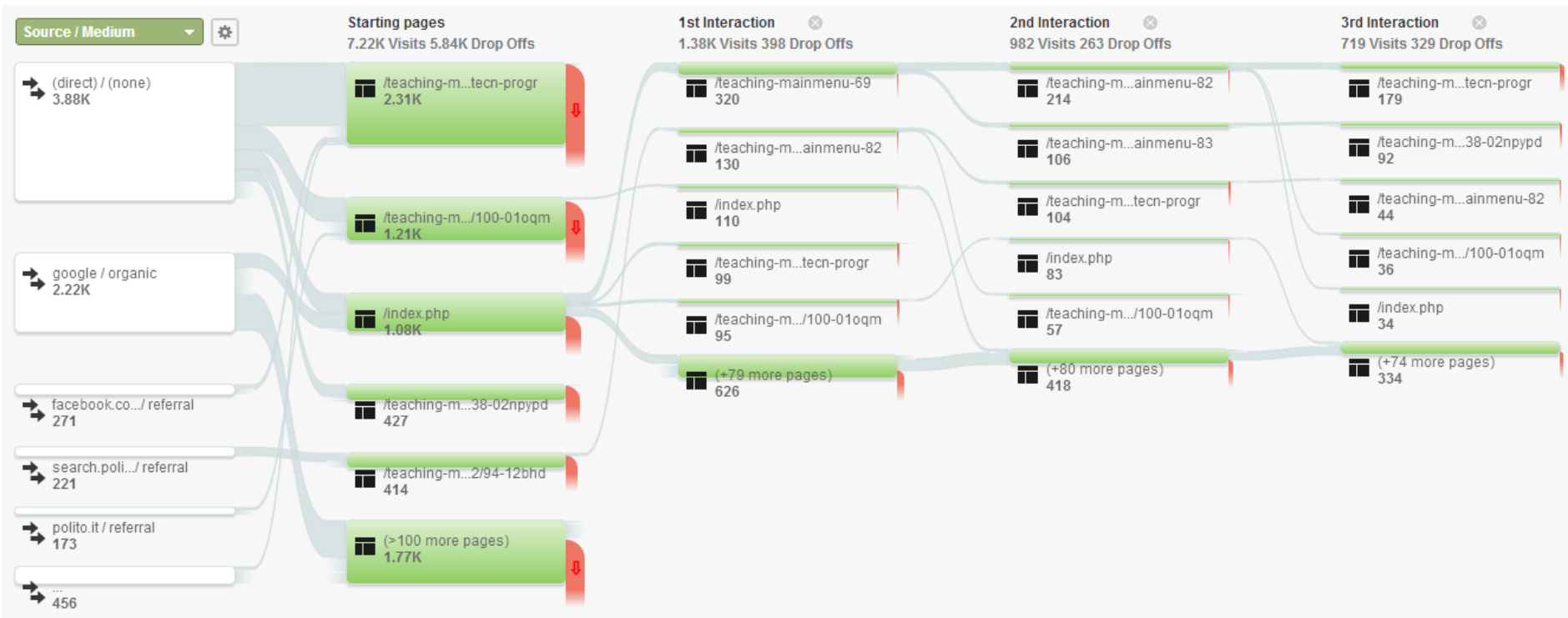
Connecting cities

- ▶ We have a water reservoir
- ▶ We need to serve many cities
 - ▶ Directly or indirectly
- ▶ What is the most efficient set of inter-city water connections?
- ▶ Also for telephony, gas, electricity, ...

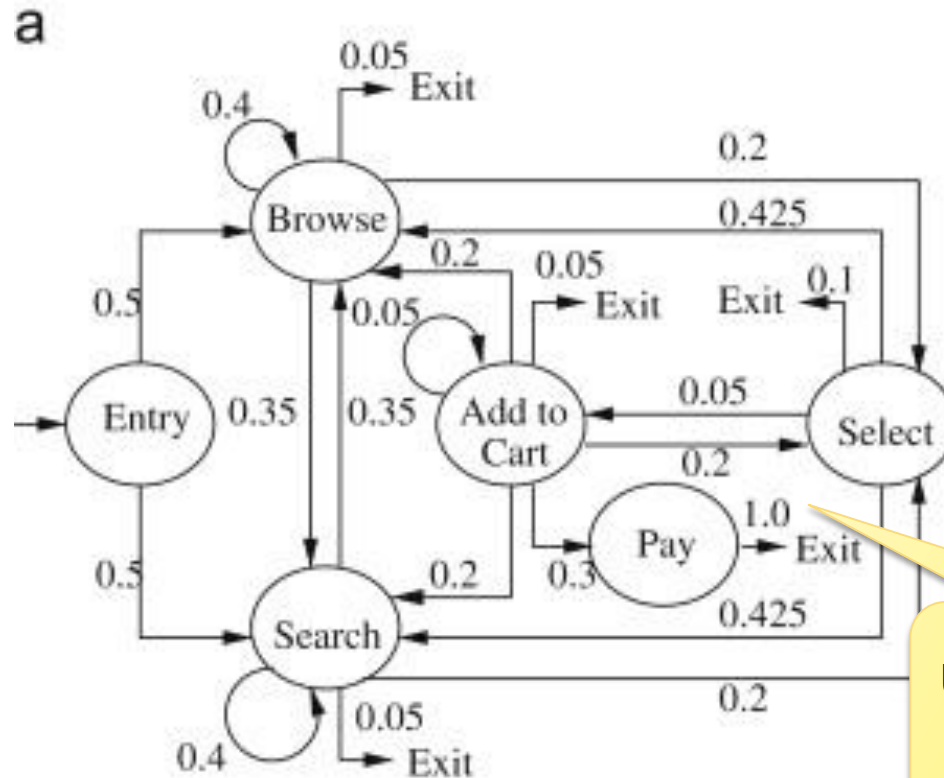
We are searching for the “minimum spanning tree”



Google Analytics (Visitors Flow)



Customer behavior



User actions encoded as frequencies

Street navigation



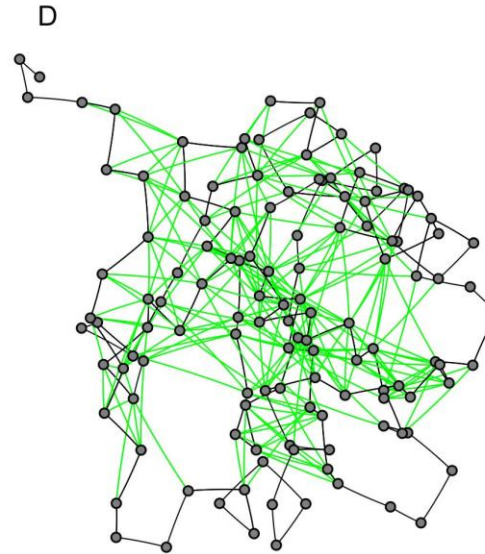
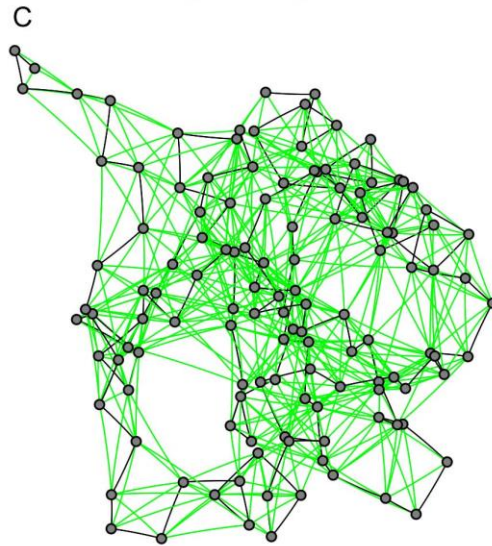
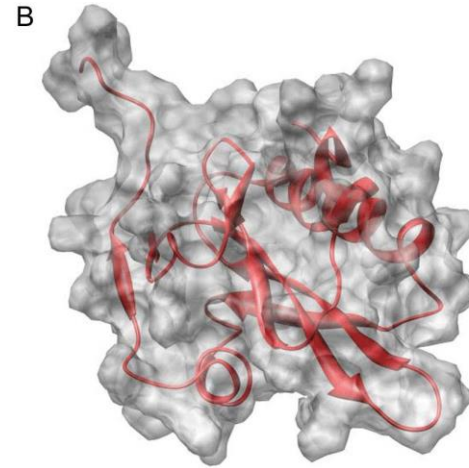
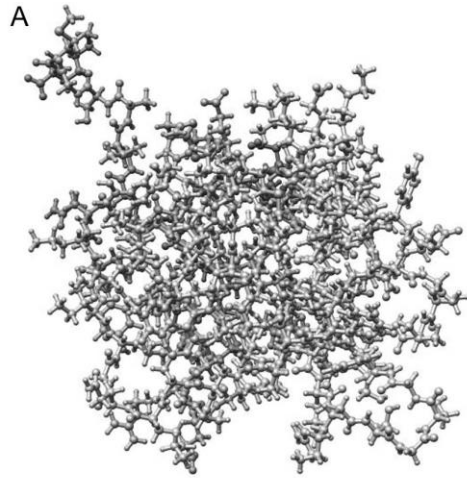
TSP: The traveling salesman problem

We must find a “Hamiltonian cycle”

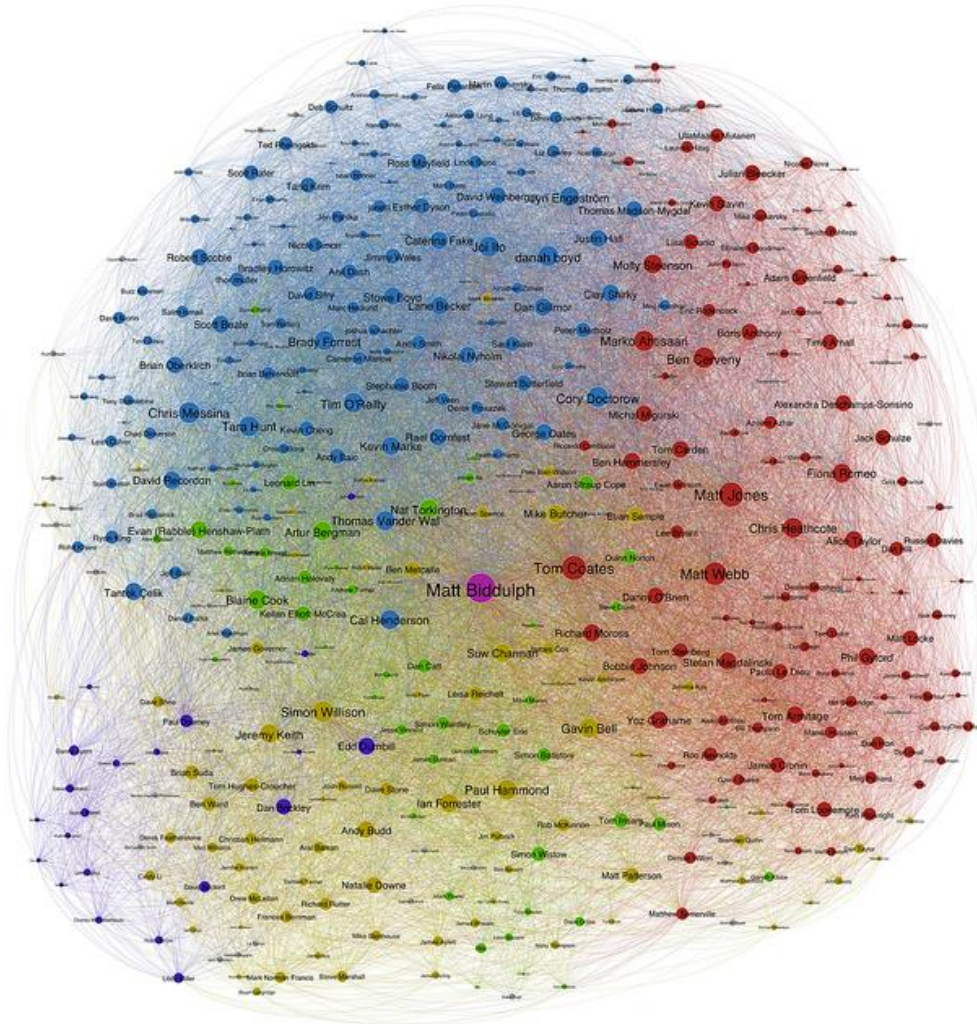
Train maps



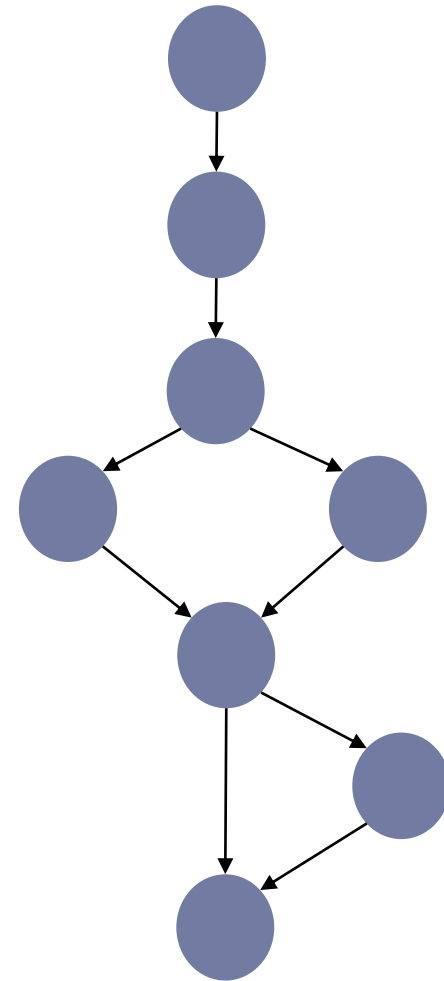
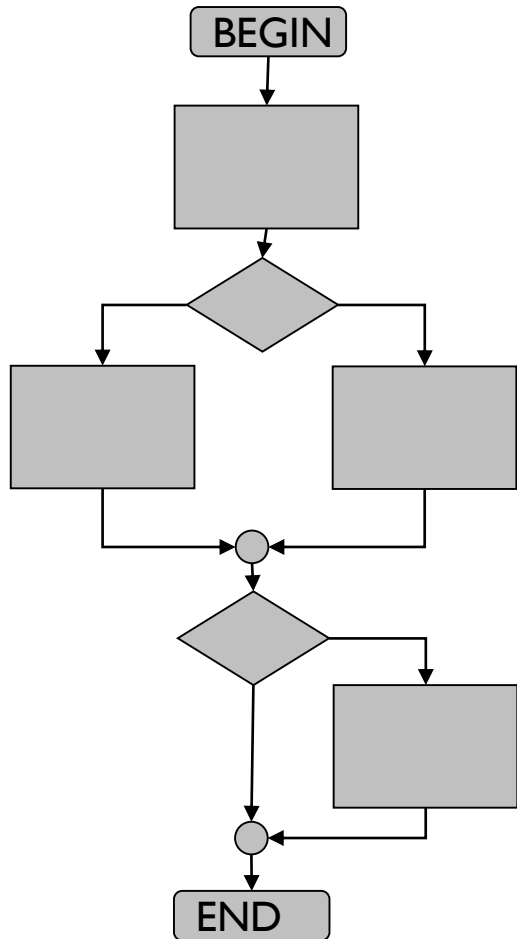
Chemistry (Protein folding)



Facebook friends








Flow chart



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